JAST
Journal of Aerospace Science and Technology

Pages: 50-60 / Vol. 16/ No. 2 /Summer- Fall 2023

Journal of Aerospace Science and Technology

www:jast.ias.ir

Scienctifie-Research Article

Integrated optimization of staging and trajectory of launch vehicles using B-spline curves

Reza Jamilnia[1],
1-Faculty of Mechanical Engineering, University of Guilan, Rasht

ABSTRACT

In this article, in order to simultaneously optimize the staging and trajectory of launch vehicles, changes are made in the structure of the trajectory optimization problem. In this approach, the flight times of all stages are considered freely and as optimization variables. During the solution and in each iteration, by using the values of the flight times in that iteration and the fuel consumption rate of each stage, the masses of the fuel and structure of the stages, and the initial and instantaneous masses of the vehicle are calculated. By minimizing the initial mass as the objective function of the integrated optimization problem, the optimal flight trajectory is obtained in the form of the optimal state and control values, and the optimal budget of the fuel and structure masses between different stages is calculated. In this article, the direct collocation method is used to implement the dynamic equations, and to approximate the variables, the B-spline curves are used. By using the B-spline curves, despite the discreteness of the relevant parameters as optimization variables, a continuous concept for the optimal solution can be created. The presented approach in this article for integrated optimization of staging and trajectory and the use of B-spline curves in the approximation of the multiphase problem with free final times can lead to reducing the initial weight of the Europa 2 launch vehicle by 30% to perform a specific mission.

Keywords: staging optimization, trajectory optimization, launch vehicles, direct collocation, B-spline curves

Introduction

Nowadays, achieving the optimal design and performance of launch vehicles to reduce the costs of space missions and increase the efficiency and capability of the vehicles has been given a lot of attention. The main task of a launch vehicle is to accelerate a payload to reach a certain velocity and position in space.

A significant part of the mass of any launch vehicle is the mass of its fuel. During the flight, due to the gradual consumption of fuel, part of the structure's mass, which is designed for primary fuel storage, becomes unused. At the same time, it causes the loss of a part of the vehicle's generated acceleration. Therefore, launch vehicles are designed as multi-stage so that when each stage is completed, that stage is separated from the vehicle, and the generated acceleration is used to accelerate the rest of the vehicle.

By staging a launch vehicle, it is possible to discard unused masses (caused by fuel

1 Assistant Professor, (corresponding author). Email: jamilnia@guilan.ac.ir

DOI : 10.22034/jast.2023.393426.1146
Submit: 17.04.2023 / Accepted: 29.07.2023
Print ISSN:1735-2134 Online ISSN: 2345-3648

consumption) in separations, and it is possible to adapt the engines and fuels of the stages to the environment and conditions in which they work. Launch vehicle designers generally use different engines and fuels for different stages to maximize the advantages of staging. This causes the optimal budgeting of fuel and structure mass among different stages to become important.

In staging optimization, the goal is to achieve the best way of budgeting fuel mass and structure mass (which depends on fuel mass) between stages so that the vehicle can launch with this combination, with minimum initial mass (which is an indicator of minimum cost and ease of launch) to reach a specified final velocity (mission velocity). Until now, extensive efforts have been made to optimize staging, which is mentioned in [1] to some common approaches in this field.

In the classic approaches presented in [1] for tandem and parallel staging, in addition to many simplifications in the case of the problem, the entire optimization process of staging is based on the ideal velocity. A launch vehicle can reach the ideal velocity when all the power of its thrust is used only for accelerating the vehicle. In reality, a part of the production thrust force is wasted due to various factors such as gravity, drag, and flight maneuvers.

Some of these losses have been considered in [2] and [3], and the optimal staging has been obtained by estimating the loss rates. This approach in [4] for reusable launch vehicles, used and in [5], has been developed to minimize the life cycle cost of this type of vehicle.

But it should be noted that it is impossible to determine the exact amount of all these losses without the vehicle's flight trajectory. By optimizing the flight trajectory, it is possible to determine a trajectory that leads to minimum loss of thrust force and, in other words, minimum fuel consumption. Therefore, to accurately optimize the staging of launch vehicles, it is necessary to determine their optimal trajectory based on fuel consumption minimization.

Some researchers have obtained a more accurate estimate of the velocity loss by finding the quasi-optimal ascent trajectories of launch vehicles in multidisciplinary design optimization structures, and they have reached optimal conceptual designs for launch vehicles with solid fuel engines [6] and hybrid engines [7]. This approach has also been used in multi-objective optimization structures [8].

In the mentioned works, quasi-optimal and sub-optimal trajectories have been used to avoid solving the very complex problem of optimizing the trajectory of multi-stage launch vehicles. Solving this problem requires using advanced optimal control methods (optimization of controllable dynamic systems) to be able to consider dynamic equations and various constraints and objective functions. With the remarkable progress and improvement of computer hardware and software in recent decades, it is possible to use complex numerical algorithms and provide various approaches in this field. Today, different approaches have been proposed to accurately solve trajectory optimization problems, some of which have been mentioned in [9].

So far, limited work has been done regarding simultaneous and accurate optimization of staging and trajectory using optimal control methods. In [10], staging and trajectory optimization problems have been solved separately and based on direct optimal control methods. An overhead program connects these problems, and by creating a repetitive cycle between two optimization programs, the overhead program can simultaneously optimize the vehicles' staging and trajectory. This approach cannot simultaneously consider the objective functions and constraints of two problems due to solving the optimization problems separately. Of course, this separate structure has advantages due to the possibility of using it in multidisciplinary design optimization structures as well as the simplicity of the solution.

This article proposes a new approach for simultaneously optimizing staging and trajectory. In this approach, by making changes in the structure of the trajectory optimization problem, it is possible to optimize the stage during the trajectory optimization in the form of a single and integrated problem. This integration causes the objective functions and constraints of two issues to be considered simultaneously, and a more optimal design is obtained than a separate approach.

In the approach used in this article, B-spline curves are used to approximate the ascent trajectories. With remarkable features such as high flexibility and formability, local behavior, and no need for continuity constraints, these curves can create continuous optimal trajectories with discrete values and provide the basis for using various nonlinear optimization methods to solve optimal control problems. These curves have previously been used in generating optimal trajectories for the descent of reentry vehicles into the atmosphere [11] and soft landing on the Moon [12] and have led to very accurate and smooth optimal trajectories.

In classical solution methods, point approximations are used to discretize and provide the possibility of using various nonlinear optimization methods, which impose a discrete nature on the optimal solution of the problem. By using the curves in the approximation of the problems, it is possible to create a continuous concept for the optimal solution of the problem while the optimization variables (coefficients or control points of the curves) are discrete. Also, since the functions of the curves are known, the values of the variables (as well as their derivatives and integrals) will be available in the entire time interval without the need for interpolation. Of course, it should be noted that the approximation of variables with curves restricts the solutions to the problem within the function of those curves. Due to their high flexibility, the use of B-spline curves significantly reduces the mentioned restriction and simultaneously allows for satisfying the strict constraints of the trajectory and boundary conditions.

The novelty of this article is combining and integrating the problems of trajectory optimization and staging optimization and using B-spline curves in the continuous approximation of state and control variables for a multiphase problem with free finite times. This approach has significantly reduced the initial weight of launch vehicles to perform a specific mission. In the numerical example presented in this article for the Europa 2 launch vehicle, this reduction was around 30%.

The remainder of this article states the governing equations of the integrated optimization problem of staging and trajectory, and the relevant optimization problem and its solution method are presented. Also, a numerical example is presented and solved to investigate the capabilities of the presented approach, and the results are compared with the separated approach of simultaneous staging and trajectory optimization [10].

Governing Equations

In this article, staging optimization is performed simultaneously by changing the trajectory optimization problem.

For optimizing the trajectory, the 3DOF point mass equations of motion of the launch vehicle in the velocity coordinate system are used [10]:

$$\dot{V} = (r\omega_E{}^2 - g)\sin\gamma + \frac{T\cos\sigma\cos\varepsilon + F_w^x}{m} \tag{1}$$

$$\begin{aligned}\dot{\chi} &= \frac{V}{r}\cos\gamma\sin\chi\tan\delta + 2\omega_E(\sin\delta - \cos\chi\cos\delta\tan\gamma) \\ &\quad - \frac{T\cos\sigma\sin\varepsilon - F_w^y}{mV\cos\gamma}\end{aligned} \tag{2}$$

$$\begin{aligned}\dot{\gamma} &= \left(\frac{V}{r} - \frac{g}{V}\right)\cos\gamma + 2\omega_E\sin\chi\cos\delta + \frac{r\omega_E{}^2}{V}\cos\gamma \\ &\quad + \frac{T\sin\sigma + F_w^z}{mV}\end{aligned} \tag{3}$$

$$\dot{r} = V\sin\gamma \tag{4}$$

$$\dot{\delta} = \frac{V}{r}\cos\gamma\cos\chi \tag{5}$$

$$\dot{\theta} = \frac{V\cos\gamma\sin\chi}{r\cos\delta} \tag{6}$$

$$\dot{Q} = 0.5\rho V^3 \tag{7}$$

The state variables in these equations are: V is the velocity, χ is the heading angle, γ is the flight path angle, r is the distance to the center of the earth, δ is the latitude, θ is the longitude, and Q is the aerodynamic heating. The control variables are: σ and ε are the angles of the thrust vector (which, in the absence of wind, are equivalent to the angle of attack α and the sideslip angle β, respectively). Also, in the above equations, m is the instantaneous mass of the vehicle, T is the thrust force, g is the acceleration of the earth's gravity, ω_E is the angular velocity of the earth, and ρ is the air density. Also, F_w^x, F_w^y, and F_w^z represent the aerodynamic forces in the velocity coordinate system.

The amount of trust force in each stage is obtained from the following equation:

$$T = \dot{m}g_0 I_{sp} \tag{8}$$

where $\dot{m}$ is the consumption rate, and I_{sp} is the specific impulse of the fuel of each stage.

To calculate the instantaneous mass of the launch vehicle (m), the masses of the fuel and the structure of all stages of the launch vehicle should be known. To perform the staging optimization during the trajectory optimization, it is necessary to make changes in calculating the instantaneous mass. For this purpose, the flight times of all stages are considered free and as optimization variables. During solving the optimization problem and depending on the values of these times (ti) they will have in each iteration, the fuel mass of that stage (Mfi) is calculated by using the fuel consumption rate of each stage:

$$M_{fi} = t_i \dot{m}_i \tag{9}$$

In launch vehicles, the mass of the structure of each stage is dependent and related to the fuel mass of that stage. So far, various equations have been proposed to express this relationship [1]. Here we use a quantity called structural efficiency. Structural efficiency (S_i) is the ratio of the mass of the structure of one stage (M_{si}) to the total mass of that stage (M_i):

$$S_i = \frac{M_{si}}{M_i} = \frac{M_{si}}{M_{fi} + M_{si}} \tag{10}$$

By using structural efficiency, the mass of the structure of each stage can be calculated from the fuel mass of that stage. Therefore, by having the flight times of the stages, it is possible to obtain all the masses of the fuel and structure of the stages.
In this article, the values of fuel consumption rate, fuel-specific impulse, and structural efficiency of the stages are assumed as fixed and specific values because these values depend on the production and operational capabilities of the manufacturers of launch vehicles.
In the mentioned approach, by considering the flight times of the stages as free and unspecified, the possibility of changing the masses of the fuel and the structure of the stages during the optimization of the trajectory is provided. Finally, after convergence and solving the integrated problem, the optimal trajectory and staging are determined simultaneously.
In the expression of the governing equations, especially those expressed to optimize the staging, corrections can be applied to make the problem more realistic. Usually, when the engine is turned off, some fuel remains in the tank or fueling system of the vehicle. Also, there is a short delay between the engine cutoff and separation and from separation to the next stage of engine start. Considering these cases in the equations can only be done by applying some fixed values (masses of remaining fuels and delay times) and does not change the structure of optimization.

The integrated optimization problem

Now, according to the discussed issues, it is possible to define the problem of integrated optimization of staging and trajectory. In this problem, the objective is to determine the optimal state and control values during the flight trajectory and flight times of the stages in such a way that the vehicle can perform its mission with the minimum initial mass and with the best mass budgeting of the fuel and structure of the stages.
Therefore, the objective function is equal to the initial mass of the launch vehicle (sum of the mass of the stages and the payload), which must be minimized and can be defined as follows:

$$J = M_0 = P + \sum_{i=1}^{N} M_{fi} + M_{si} \tag{11}$$

where M_0 is the initial mass of the vehicle, P is the mass of the payload, and N is the number of stages. In this article, the mass of the payload and the number of stages are assumed to be constant, according to the defined design problem.

In the integrated optimal control problem, the governing equations stated in the previous section, including the equations of motion and the equations related to determining the mass of the fuel and the structure of the stages, are considered state equations and path constraints. For this problem, various other constraints, such as maximum dynamic pressure and maximum aerodynamic heating, can also be defined and applied.

The initial and final conditions of the trajectory can be defined according to the launch vehicle's mission. In this article, the optimization process starts from the end of the vertical flight of the launch vehicle and ends when it reaches the destination orbit. Therefore, the initial conditions are determined according to the conditions of the launching vehicle after the vertical flight, and the final conditions are determined according to the conditions of the vehicle when entering the destination orbit.

In designing the launch vehicles' trajectory, one or more free flights are usually expected between stages. These free flights help the vehicle to approach some final conditions without fuel consumption. If the time of this free flight is not chosen correctly and optimally, it can cause velocity loss. As a result, if free flight is used along the flight trajectory, it is necessary to consider its time as an optimization variable. In the defined integrated problem, apart from the free flight time, the time of all stages is also free and unspecified.

Solution Method

The exact solution of nonlinear optimal control problems with multiple point constraints (boundary conditions), path constraints, and complex dynamic equations and models is complicated. For this purpose, researchers have proposed various methods based on converting optimal control problems into nonlinear optimization problems. Some of these methods (such as the indirect and direct shooting methods) maintain the dynamics and integration, but others eliminate the dynamics and integration by full discretization. Considering the complexity and time-consuming process of accurate integration in the structures of optimization problems, the methods based on dynamics elimination are more acceptable and faster [13].

Direct Collocation

The direct collocation method is one of the methods based on dynamics elimination, in which by defining the time nodes during the time interval of the problem and collocating the state and control variables as a sequence of discrete values, the dynamics of the problem are fully discretized, and the integration process is eliminated. In this case, instead of time-dependent functions of state and control variables, discrete values of state and control at the time nodes are considered as optimization variables. After discretizing the problem and eliminating the dynamics and time, redefining and solving the problem as a nonlinear optimization problem and achieving the optimal state and control values in the nodes is possible.
In the classical direct collocation method, the discrete values of the state and control at the time nodes are used to approximate the differential and integral expressions. In this situation, to increase the accuracy of approximation and solution, it is necessary to define a very large number of time nodes (and, consequently, a very large number of optimization variables). This increases the dimensions of the nonlinear optimization problem and prolongs its solution time [13].
Unlike the classical approach, B-spline curves in this article approximate the state and control variables. In this way, the B-spline curves' control points are considered as discrete optimization variables. Due to the specificity of the functions of the B-spline curves and their derivatives and integrals, the role of time nodes in approximating differential and integral expressions is practically lost, and time nodes are only used to apply equations of state and point and path constraints. In this case, there is no need to significantly increase the number of time nodes to increase the accuracy of the solution. With a limited number of time nodes and smaller dimensions, the nonlinear optimization problem can be solved accurately and quickly.

B-spline Curves

B-spline curves are helpful mathematical tools that can be used to approximate a continuous concept with discrete values. Although the coefficients or control points are discrete values, the curves produce a continuous output by being placed in the curve's function. This feature can be useful in the numerical solution methods of optimal control problems because, as mentioned earlier, the general approach in these methods is to convert an optimal control problem (with dynamic equations and time-dependent functions) into a nonlinear optimization problem (with algebraic equations and time-independent functions).
For choosing curves to approximate variables, polynomials are usually the first choice. But polynomials are not a good choice because polynomials cannot approximate various trajectories. Even if we use high degrees, not only a better approximation is not achieved, but fluctuating trajectories are obtained, which are undesirable in most cases. The way to solve this problem is to use several connected polynomials with low degrees. Although various trajectories can be approximated in this case, other problems arise. At the points where the curves are connected, constraints should be considered to maintain the conditions of continuity and derivability. This imposes additional constraints on the problem. In addition to the mentioned problems, polynomials have other disadvantages. In polynomials, the coefficients' values and their importance are not equal. As a result, the ranges of their changes are different. Polynomials do not behave locally, and changing a coefficient shifts the whole curve. Finally, it can be said that polynomials are not suitable curves for approximating variables.
B-spline curves are a continuous collection of several Bezier curves, widely used in engineering

and computer graphics due to their unique features. These curves can create flexible and controlled trajectories without applying continuity constraints between the constituent Bezier curves [14].
The general form of these curves is as follows:

$$x(t)=\sum_{i=0}^{n} B_{i,k}(t)C_i \qquad t_0 \le t \le t_f \tag{12}$$

According to the above equation, B-spline curves are composed of two parts of basis functions ($B_{i,k}(t)$) and control points (C_i). In order to calculate the basis functions of a B-spline curve, the number of Bezier curves and their degrees should be determined in proportion to the complexity of the expected trajectory for the approximated variable. Then, the time interval is divided according to the number of Bezier curves. The time points of this division are called nodes, and by putting them together in a vector, the node vector (τ) is defined:

$$\boldsymbol{\tau}=[t_0,t_1,\ldots,t_{n-1},t_n] \tag{13}$$

In the node vector, a time value may be repeated several times in a row, which is called multiplicity. The difference in the order of the curve (k_i) and the number of corresponding nodes (m_i) specifies the degree of smoothness (s_i). It should be mentioned that the order of the curve is equal to the degree of the curve plus one:

$$s_i = k_i - m_i \tag{14}$$

The degree of smoothness indicates the level of continuity in the node. Therefore, the continuity and derivability of the curve at the beginning and end of Bezier curves can be established in the definition of the node vector by repeating the time values of the nodes, and to apply continuity and derivability between connected Bezier curves does not need to define additional constraints.
With the order of the curves (k), number of curves (l), and smoothness of nodes (s), the number of control points (P) can be specified:

$$P = l(k-s)+s \tag{15}$$

With the mentioned items being known, it is possible to calculate the basis functions by using the following recursive relations:

$$B_{i,0}(t)=\begin{cases}1 & \text{if } t_i \le t \le t_{i+1}\\ 0 & \text{otherwise}\end{cases} \tag{16}$$

$$B_{i,k}(t)=\frac{t-t_i}{t_{i+k+1}-t_i}B_{i,k-1}(t)+\frac{t_{i+k}-t}{t_{i+k}-t_{i+1}}B_{i+1,k-1}(t) \tag{17}$$

Control points in a B-spline curve are the points around the curve that form the curve, and in the true sense of the word, they are the curve's control points. These control points, like coefficients of polynomials, are discrete values that produce a continuous concept [15]. In approximating state and control variables, these control points can be considered as optimization variables. In Figure 1, a B-spline curve with its control points is shown.

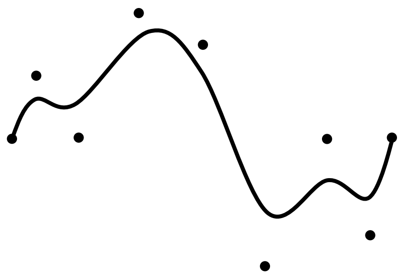

Figure 1. A B-spline curve with its control points

B-spline curves have a completely local behavior. By changing one of the control points, depending on the degree of the curve, only the shape of the curve in the neighborhood of that control point changes, and the rest of the curve remains unchanged. Also, the range of changes of control points in B-spline curves is the same and almost equal to the range of changes of the approximated variable. Suppose the B-spline curves are used to approximate the variables of the trajectory optimization problem. In that case, the possibility of accurate calculation of the time derivatives of the variables is easily provided due to the specificity of the time derivatives of the B-spline curve functions.
B-spline curves seem to be appropriate curves for approximating state and control variables in optimal control problems due to the desirable approximation of complex trajectories, continuous conceptual expression with discrete control points, no need to define continuity constraints, completely local behavior and the same range of changes of control points with approximated variables.

Nonlinear Programming

In this article, using the direct collocation method, the flight time interval of the stages (all of which are assumed to be free and unspecified) are divided into several equally spaced time intervals. At the beginning and end points of these subintervals (time nodes or collocation points), state equations and constraints of the problem should be applied. To apply these equations in the mentioned points, the state and control variables should be approximated with appropriate B-spline curves (in terms of flexibility and formability and the necessary level of continuity and derivability). Suppose the control points of the selected B-spline curves and the free times of the stages are considered as optimization variables, in each iteration of the optimization, depending on the values of the control points in that iteration. In that case, it is possible to obtain the approximate values of the state and control at the collocation points from the B-spline curves and apply the state equations and constraints in the form of equality and inequality constraints to the optimization problem. Also, based on the state and control values at the collocation points, it is possible to calculate the value of the objective function, which is the initial mass of the launch vehicle in the given problem.

This nonlinear optimization problem can be solved by various optimization methods based on gradient or population. One of the most accurate methods that can be used is nonlinear programming, which is a collection of gradient-based methods and corrective algorithms for line search and trust region. In this article, in order to solve the problem of nonlinear optimization, an open-source nonlinear programming solver named IPOPT is used. Using the primal-dual interior point method, this software can solve large-scale nonlinear programming problems with high accuracy and speed [16].

Numerical Example

This section presents a numerical example to demonstrate the proposed approach's capabilities. In this example, the specifications of a four-stage launch vehicle (Europa 2) have been used [17]. The values of fuel consumption rate, fuel specific impulse, and structural efficiency of the stages are presented in Table 1.

Table 1. The specifications of Europa 2 [10]

Stage	$\dot{m}$(kg/s)	I_{sp} (s)	S
I	549.1	280.1	0.07
II	95.8	280.9	0.22
III	7.9	300.2	0.25
IV	15.2	275.5	0.13

The prescribed mission for this launch vehicle is to deliver a 387 kg payload to the perigee point of a geosynchronous transfer orbit. The apogee and perigee points of chosen transfer orbit for this mission are 35950 km and 250 km, respectively, with a 7 deg inclination. The vertical flight time is 10 s, and the latitude and longitude of the launch site (Korou, French Guyana) are 5.24 deg and -52.78 deg, respectively. A free flight phase is prescribed between the 3rd and 4th stages.

The initial and final conditions of the trajectory for the prescribed mission are presented in Table 2. In this table, the conditions related to the vehicle's distance from the center of the earth (r) are expressed in the form of altitude above the earth's surface (h), which is obtained by subtracting the average radius of the earth from the radial distance.

Table 2. Boundary conditions for the mission

State variable	Initial value	Final value
V (m/s)	60.0	9718.2
χ (deg)	free	97.3
γ (deg)	90.0	0.0
h (km)	0.3	250.0
δ (deg)	5.24	0.0
θ (deg)	-52.78	free
Q (j)	۳۵	free

In order to solve this example, the state and control variables in each stage are approximated with a 3rd-degree B-spline curve consisting of 6 Bezier curves with a continuity level of 3. This curve will have 9 control points. Therefore, to approximate each state and control variable in each stage, 9 optimization variables will be created. The selected values are the values that satisfy the necessity in terms of formability and continuity of the problem.

Also, in this example, 11 collocation points (time nodes) are considered to apply state equations and point and path constraints for each phase of the vehicle's flight. These points divide the time interval of each phase into 10 equal subintervals.

The results of the integrated staging and trajectory optimization for this launch vehicle are presented in Table 3 (optimal staging). The ideal, actual, and wasted velocities of stages are listed in Table 4.

In general, about 12% of the ideal velocity was wasted due to the presence of waste factors, and the vehicle was able to produce an actual velocity of 9658.2 m/s, which, with an initial velocity of 60 m/s, the vehicle was able to achieve mission velocity of 9718.2 m/s. As can be seen, free flight has caused a slight velocity loss, which is acceptable compared to the overall advantage of free flight in reaching some final conditions.

Table 3. Optimal staging (integrated)

Stage	t (s)	M_f (kg)	M_s (kg)
I	117.1	64309.1	4840.5
II	42.0	4025.3	1135.4
III	194.0	1532.9	511.0
Free flight	391.2	0.0	0.0
IV	63.0	957.0	143.0
Total	807.3	70824.3	6629.9

Table 4. The ideal, actual, and wasted velocities (integrated)

Stage	ΔV_{id} (m/s)	ΔV_{act} (m/s)	ΔV_{loss} (m/s)
I	4806.0	3788.5	1017.5
II	1713.4	1631.3	82.1
III	1676.4	1535.8	140.6
Free flight	0.0	-78.0	78.0
IV	2787.0	2780.6	604
Total	10982.8	9658.2	1324.6

Figure 2 shows the time histories of state and control variables (optimal trajectories and controls). In this figure, the time interval of each flight phase is marked with vertical lines.

As shown in Figure 2, the launch vehicle strongly tends to fly horizontally to minimize fuel consumption. Because, in this case, a considerable part of the gravity force is neutralized by the centrifugal force, and thrust force is mostly used to accelerate the vehicle.

According to Figure 2, due to the sudden decrease in the weight of the vehicle during the separation of the stages, as well as the changes in the thrust force corresponding to the fuel and engine of each stage, changes in the slope of the velocity diagram can be seen. Also, the separation of the stages can be observed in this diagram.

As can be seen in the diagrams of angles of attack and sideslip (or thrust vector angles), during free flight, the control commands are zero because, at this phase of the flight, the change of thrust vector angles due to the absence of thrust force will not affect the movement of the vehicle.

Based on the diagrams presented in Figure 2, the initial and final conditions determined for all state variables have been applied, and the vehicle at the end of the trajectory has obtained the necessary conditions to enter the geosynchronous transfer orbit.

Comparison of the integrated approach with the separated approach

In the separated approach presented in [10], the simultaneous optimization process of staging and trajectory is done separately, and an overhead program by creating a connection between the inputs and outputs of the optimization programs of staging and trajectory and using the Regula-Falsi method can determine the optimal staging and the optimal trajectory.

In this approach, first, the overhead program considers an ideal velocity as an initial guess according to the mission velocity. The staging optimization program obtains the optimal staging so the vehicle achieves the ideal velocity with the minimum initial mass. Then, using this optimal staging (masses of fuel and structure), the trajectory optimization program determines the optimal trajectory so that the vehicle can bring the maximum payload mass to the desired final conditions. The overhead program obtains the new ideal velocity by comparing the maximum payload mass obtained from the trajectory optimization with the mission payload mass. The staging optimization program provides the optimal staging using the new ideal velocity, and the trajectory optimization program obtains the maximum payload mass using the new optimal staging. This cycle is repeated until the maximum payload mass equals the mission payload mass. If we solve the example presented in this article with a separate approach, we will reach the results presented in Table 5 and Table 6.

Table 5. Optimal staging (separated)

Stage	t (s)	M_f (kg)	M_s (kg)
I	117.2	64357.4	4844.1
II	41.8	4005.2	1129.7
III	206.3	1629.7	543.2
Free flight	290.9	0.0	0.0
IV	71.3	1084.2	162.0
Total	727.5	71076.5	6679.0

Table 6. The ideal, actual, and wasted velocities (separated)

Stage	ΔV_{id} (m/s)	ΔV_{act} (m/s)	ΔV_{loss} (m/s)
I	4765.6	3753.7	1011.9
II	1636.7	1554.2	82.5
III	1645.4	1483.4	162.0
Free flight	0.0	-63.0	63.0
IV	2945.4	2929.9	15.5
Total	10993.1	9658.2	1334.9

The comparison of the results of solving the presented example with integrated and separated approaches, despite the structural differences between the two approaches, shows the closeness of the results to each other and indicates the superiority of the integrated approach. The integrated approach with 301.3 kg (0.4 percent) lower initial mass than the separated approach can accomplish the prescribed mission and lead to a more optimal staging and trajectory.

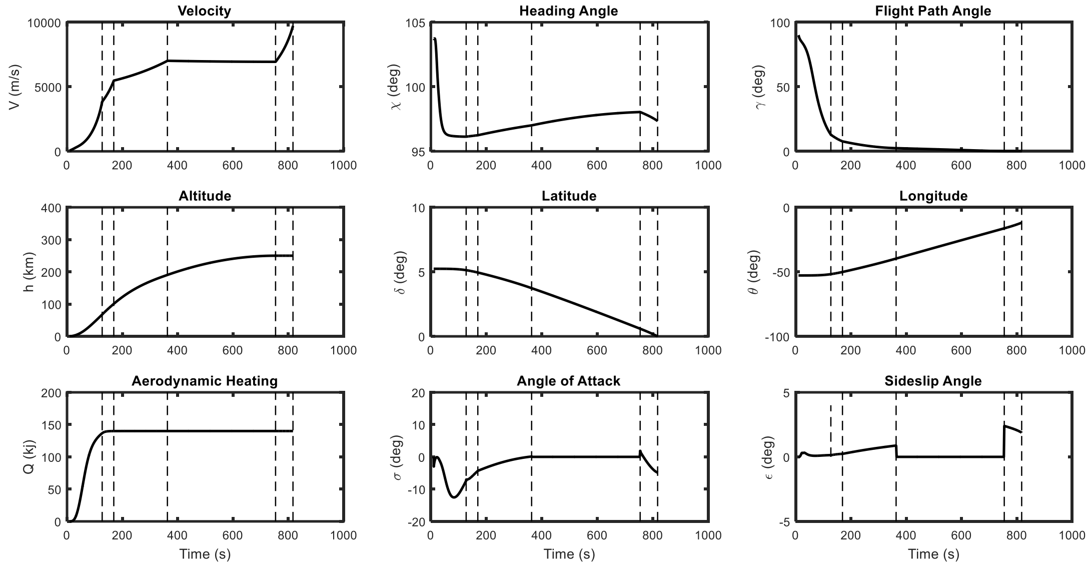

Figure 2. Time histories of state and control variables.

Conclusion

In this article, integrated optimization of staging and trajectory was presented. Considering the objective functions and the effective constraints of both staging and trajectory optimization problems, the presented approach can provide the optimal combination of staging and trajectory for launch vehicles.

In this approach, the optimal staging is determined based on the actual velocity of the launch vehicle. In contrast, in previous works, the staging optimization was generally based on the ideal velocity. In this approach, all the forces on the launch vehicle and maneuvers of the trajectory are affected in the staging, and even the effects of the free flight phase are considered.

Due to its fully numerical implementation, the solution method used in this article has a flexible and independent structure from the state equations. It provides the possibility to change, modify or improve the equations without affecting the structure of the solution method. This feature makes it possible to investigate various missions with different scenarios.

Due to the use of advanced gradient methods in nonlinear programming, the speed of solving the integrated problem, despite the great complexity and the free final times applied in the problem (which causes the convergence ability to decrease), is very significant.

Considering the desired performance of staging optimization in the form of trajectory optimization, the possibility of optimizing other design parameters also seems possible with the proposed approach.

References

[1] E. Keith, *The New Rocket Science*, Lulu Enterprises Incorporated, 2010.

[2] G. Palaia, , M. Pallone, M. Pontani, “Ascent Trajectory Optimization and Neighboring Optimal Guidance of Multistage Launch Vehicles”, *Springer Optimization and Its Applications*, Vol. 144, 2019, pp. 343-371.

[3] A. Koch, “Optimal Staging of Serially Staged Rockets with Velocity Losses and Fairing Separation”, *Aerospace Science and Technology*, Vol. 88, 2019, pp. 65-72.

[4] B. Jo, J. Ahn, “Optimal Staging of Reusable Launch Vehicles Considering Velocity Losses”, *Aerospace Science and Technology*, Vol. 109, 2021.

[5] B. Jo, J. Ahn, “Optimal Staging of Reusable Launch Vehicles for Minimum Life Cycle Cost”, *Aerospace Science and Technology*, Vol. 127, 2022.

[6] Q. Zhang, Z. Xu, “Autonomous Ascent Guidance with Multiple Terminal Constraints for All-Solid Launch Vehicles”, *Aerospace Science and Technology*, Vol. 97, 2020.

[7] Y. Liu, X. Li, P. Wang, X. Zhang, "Multi-Objective, Multi-Disciplinary Design Optimization and Multi-Attribute Evaluation of Hybrid Rocket Motors Used for Manned Lunar Lander", Aerospace, MDPI, Vol. 10, No. 3, 2023.

[8] K. Wang, B. Zhang, "Multiobjective Trajectory Optimization for A Suborbital Spaceplane Using Directed Search Domain Approach", *Aerospace Science and Technology*, Vol. 77, 2018, pp. 713-724.

[9] J. Betts, "Survey of Numerical Methods for Trajectory Optimization", *Journal of the Guidance, Control and Dynamics*, Vol. 21, No. 2, 1998, pp. 193-207.

[10] R. Jamilnia, "Multidisciplinary Design Optimization of Multistage Launch Vehicles", M.Sc. Thesis, Amirkabir University of Technology, 2005 (in Persian).

[11] R. Jamilnia, "Optimal Guidance of a Reentry Vehicle Based on Online Trajectory Optimization", *Journal of Aerospace Science and Technology (JAST)*, Vol. 13, No. 1, 2020, pp. 98–105.

[12] R. Jamilnia, "Optimal Trajectory Design for Soft Landing on the Moon by Using Differential Flatness", *Modares Mechanical Engineering*, Vol. 17, No. 10, 2018, pp. 9–19.

[13] J. Betts, *Practical Methods for Optimal Control Using Nonlinear Programming*, SIAM, 2010.

[14] C. De Boor, *A Practical Guide to Splines*, New York: Springer-Verlag, 2001.

[15] K. Höllig, J. Hörner, *Approximation and Modeling with B-Splines*, SIAM, 2013.

[16] A. Wächter, 2008, *Introduction to IPOPT*, Carnegie Mellon University, 2008.

[17] B. Harvey, 2003, *Europe's Space Programme: To Ariane and Beyond*, Springer, 2003.