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# Analysis of Laminar Film Condensation on a Vertical Plate

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*This paper concerns the condensation phenomenon on a vertical plate. Theoretical background has been explained using some dimensionless parameters, which help to understand the behavior of this phenomenon in more details. The scale up of boundary layer theory has been used to transfer the governing equations to ordinary differential equations. Next, the governing equations have been solved numerically using the shooting method. A computer program has been developed to solve these equations and the heat transfer coefficient of a condensate film for a wide range of physical properties has been obtained, presented graphically, and compared with other available literatures.*

## NOMENCLATURE

| | |
|---|---|
| $A$ | Matrix of constant coefficients |
| $Ar$ | Archimedes number |
| $C_p$ | Specific heat at constant pressure |
| $C_1, C_2$ | Functions of $(\eta, \mathrm{Pr})$ |
| $c$ | Constant |
| $F, G$ | Functions of $(\eta, \mathrm{Pr})$ |
| $g$ | Gravity acceleration, $(\mathrm{m/s^2})$ |
| $h$ | Convection coefficient, $(\mathrm{W/m^2K})$ |
| $h_{fg}$ | Enthalpy, (J/Kg.K) |
| $Ja$ | Jacob number |
| $k$ | Thermal conductivity, (W/m.K) |
| $L$ | Plate length, (m) |
| $m, n$ | Constant powers |
| $Nu$ | Nusselt number |
| $Pr$ | Prandtl number |
| $Q$ | Discrepancy vector |
| $Q$ | Heat transfer, (J) |
| $T$ | Temperature, (K) |
| $u$ | x-velocity component, (m/s) |
| $V$ | vector of unknown boundary conditions |
| $v$ | y-velocity component (m/s) |
| $x$ | vertical coordinate |
| $y$ | horizontal coordinate |
| $Y$ | vector of unknown variables |

### Greek Letters

| | |
|---|---|
| $\alpha$ | Thermal diffusivity, $\mathrm{m^2/2}$ |
| $\alpha$ | Jacobian matrix |
| $\theta$ | nondimensional temperature |
| $\eta$ | similarity parameter |
| $\psi$ | stream function |
| $\delta$ | average variation |
| $\delta$ | variation of a vector |
| $\delta$ | thickness of the condensate layer, m |
| $\delta_t$ | thickness of the thermal boundary layer, m |
| $\rho$ | density, $\mathrm{Kg/m^3}$ |
| $\mu$ | dynamic viscosity, Kg/m.s |
| $\nu$ | kinematic viscosity, $\mathrm{m^2/s}$ |

### Subscripts

| | |
|---|---|
| $c$ | natural scale |
| $i, j$ | matrix indices |
| $L$ | plate length |

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| | |
|---|---|
| $n$ | Nusselt theory result |
| $s$ | surface |
| $sat$ | saturated |
| $v$ | vapor |
| $\delta$ | boundary layer edge |

**Superscripts**

| | |
|---|---|
| $*$ | dimensionless parameter |
| $new/old$ | value at new/old iteration |

## INTRODUCTION

When the temperature difference between a fluid and a wetted plate is sufficiently large, the fluid on the surface experiences a change of phase. The heat transfer analysis of film condensation is an important area in the design and performance of fins in heat exchangers and other devices. The first reported research and experiment about condensation phenomena was from Nusselt [1].He simplified the phenomenon by ignoring energy convection and acceleration terms. After Nusselt, many researchers continued his study and made some improvements it. Rohsenow's formulation contains the effect of energy convection, but still ignores the acceleration terms [2]. Sparrow & Gregg expanded the previous works using boundary layer theory and similarity method [3]. In this way, they converted the governing equations to ordinary differential equations (O.D.E) and explained the properties of condensation on some parameters such as Jacob and Prandtl number. They showed that if the Prandtl number is large, the influences of the inertial terms are not important. The importance of such results has been well known and documented in Reference [4], extended by Koh et al. [5], Koh [6] and Chen [7] and more recently, in [8]. Experimental results on film condensation have been correlated by Chen et al. [9]. Recently, Mendez et al. [10] studied the conjugate heat transfer condensation process of saturated vapor and the effect of longitudinal heat conduction in the thermal thick wall regime was taken into account. They used perturbation methods and the boundary layer description.

The main purpose of the previous works has been to determine the basic characteristics of condensate film. In the present work, we try to develop the work of Sparrow and Gregg, focusing on more details using dimensionless equations and parameters. To achieve this aim, some scales have been used and a computer code using shooting method is developed to calculate the numerical values. Finally, the results for a wide range of physical properties are investigated.

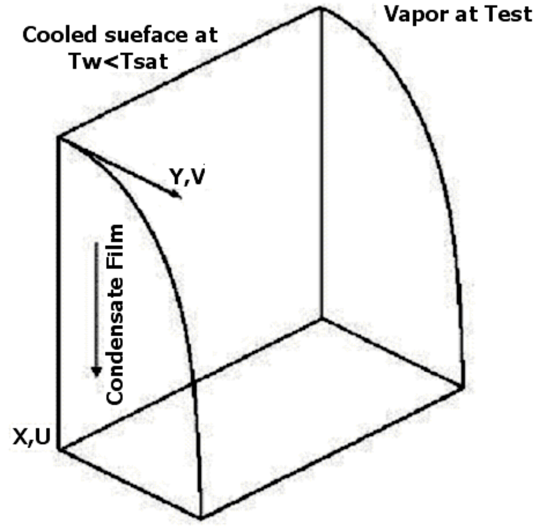


**Figure 1.** Physical model on coordinate system

## MATHEMATICAL CONSIDERATION

The physical problem contains a vertical plate that has been fixed in a body of pure vapor. The plate surface temperature is taken to be uniform ($T_w$). The vapor is at saturation temperature ($T_{sat} > T_w$), and thus, a continuous laminar film of condensate has formed on the plate, which runs downward. A schematic of coordinate system on the physical model is shown in Figure 1. It is also assumed that the velocity of vapor towards the condensate layer has no effect on the condensation film, and that viscous dissipation can be neglected.

### Conservation Laws

The valid conservation equations for this physical problem in the above coordinate system can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1a}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = g\left(\rho - \rho_v\right) + \mu\frac{\partial^2 u}{\partial y^2} \tag{1b}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} \tag{1c}$$

where $\rho_v$ is the vapor density, and other properties are those of condensate.

The equation for shear stress at plate surface is as follow:

$$\rho u\frac{\partial}{\partial x}\int_0^\delta u\,dy = -\mu\left.\frac{\partial u}{\partial y}\right|_{y=0}$$

We assumed that there is no shear stress on the boundary layer edge. Also its temperature is the saturated temperature. No-slip boundary condition was applied to the plate and the plate temperature was set to be constant. Therefore, the boundary conditions for this problem may be written as:

$$\text{at } y=0 \left\{ \begin{array}{c} u=0 \\ v=0 \\ T=T_s \end{array} \right\}; \quad \text{at } y=\delta \left\{ \begin{array}{c} \frac{\partial u}{\partial y}=0 \\ T=T_{sat} \end{array} \right\} \tag{2}$$

The thickness of the condensate layer $(\delta)$ , which is a function of $x$ , will be determined later.

### Dimensionless Equations

To change the governing equations into dimensionless form, the dimensionless parameters are defined as follows:

$$u^*=\frac{u}{u_c}; v^*=\frac{v}{v_c}; T^*=\frac{T-T_{sat}}{T_s-T_{sat}}$$

$$y^*=\frac{y}{\delta}; x^*=\frac{x}{L} \tag{3}$$

Which incorporate the natural scales designated by subscript $c$ . The balance of heat transfer in condensate film results in:

$$\alpha C_p\frac{T_{sat}-T_s}{\delta}=v_c h_{fg} \tag{4}$$

Using Jacob number which is defined as:

$$Ja=C_p\frac{T_{sat}-T_s}{h_{fg}} \tag{5}$$

and substituting from equations (3) and (4) to governing equations (1), dimensionless forms of equations are:

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \tag{6a}$$

$$\frac{Ja}{\mathrm{Pr}}\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=1+\frac{\partial^2 u^*}{\partial y^{*2}} \tag{6b}$$

$$Ja\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)=\frac{\partial^2 T^*}{\partial y^{*2}} \tag{6c}$$

### Similarity Transformation

In terms of the stream function, $(\psi)$ which is defined as:

$$u=\frac{\partial \psi}{\partial y}$$

$$v=-\frac{\partial \psi}{\partial x}$$

the governing equations (1) become:

$$\left(\frac{\partial \psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial \psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right)=\frac{g(\rho-\rho_v)}{\rho}+\nu\frac{\partial^3\psi}{\partial y^3} \tag{7a}$$

$$\left(\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y}\right)=\alpha\frac{\partial^2 T}{\partial y^2} \tag{7b}$$

Next, similarity transformation will be used to reduce the above equations to ordinary differential equations. First the following scales are defined:

$$u\approx\alpha\left(\frac{k\nu}{gC_p(\rho-\rho_v)x^3}\right)^{-\frac{1}{2}}G(\eta,\mathrm{Pr})$$

$$\delta_t\approx\left(\frac{k\nu}{gC_p(\rho-\rho_v)x}\right)^{\frac{1}{4}} \tag{8}$$

then, the similarity parameters are defined as follows:

$$\eta=\frac{y}{\delta_t}=c.y.x^{-\frac{1}{4}}$$

$$c=\left[\frac{gC_p(\rho-\rho_v)}{4\nu k}\right]^{\frac{1}{4}} \tag{9}$$

where $\delta_t$ is the thickness of the thermal boundary layer. Using the definition of $\psi$ , after some simplifications, there is:

$$\psi=4\alpha.c.x^{\frac{3}{4}}F(\eta,\mathrm{Pr}) \tag{10}$$

and dimensionless temperature $\theta$ is defined as follows:

$$\theta(\eta)=\frac{T-T_{sat}}{T_s-T_{sat}} \tag{11}$$

Substituting the similarity parameters in equations (7a) and (7b) one can get to ordinary differential form of governing equations:

$$\dddot{F}+\frac{1}{Pr}[3F\ddot{F}-2\dot{F}]+1=0 \tag{12a}$$

$$\ddot{\theta}+3F\dot{\theta}=0 \tag{12b}$$

and the boundary conditions equation (2) becomes:

$$\text{at } \eta=0\left\{\begin{array}{c} F=0 \\ \dot{F}=0 \\ \theta=1 \end{array}\right\}; \quad \text{at } \eta=\eta_\delta\left\{\begin{array}{c} \ddot{F}=0 \\ \theta=0 \end{array}\right\} \tag{13}$$

where $\eta_\delta$ is the value of $\eta$ , at interface of phases $y=\delta$ To determine the value of $\eta_\delta$ , an overall balance of energy on condensate film can be written as follows:

$$\int_0^X k\left(\frac{\partial T}{\partial y}\right)_{y=0}dx=\int_0^\delta \rho u h_{fg}dy+\int_0^\delta \rho u C_p(T_{sat}-T)dy \tag{14}$$

where the terms in this equation are heat transferred from condensate to the plate, energy liberated as latent heat, and energy liberated by sub-cooling of the condensate, respectively. In terms of similarity variables, equation (14) can be re-written as:

$$Ja=\frac{C_p(T_{sat}-T_s)}{h_{fg}}=-3\frac{F(\eta_\delta)}{\dot{\theta}(\eta_\delta)} \tag{15}$$

where $Ja$ is called Jacob number and $\dot{\theta}(\eta_\delta)$ and $F(\eta_\delta)$ are the values of $d\theta/d\eta$ and $F$ at $\eta = \eta_\delta$ , respectively [3]. With respect to the boundary conditions in equation(13), the treatment of film condensation can be analyzed by solving the equations (12) and (15). However, two out of five boundary conditions are defined at $\eta = \eta_\delta$ , and three other at origin.

## METHOD OF SOLUTION

First, equations (12) with boundary condition (13) transferred to a set of ODE in the form of:

$$\dot{Y} = AY \tag{16}$$

and:

$$Y^T = \{\ F \quad \dot{F} \quad \ddot{F} \quad \theta \quad \dot{\theta}\ \}$$

$$Y^T(0) = \{\ 0 \quad 0 \quad \ddot{F}(0) \quad 1 \quad \dot{\theta}(0)\ \}$$

$$Y^T(1) = \{\ F(1) \quad \dot{F}(1) \quad 0 \quad 0 \quad \dot{\theta}(1)\ \} \tag{17}$$

where Y is a vector of five by one and A is a square matrix. The initial conditions of the governing O.D.E. equations (12) are not entirely presented at one point. Thus, the problem we have to solve is a two-point boundary value problem. There are two distinct classes of numerical methods for solving such problems [11]. In this paper, we use the shooting method to solve equations (12), Where, two values for the dependent variables at the beginning of the domain are guessed, then integrated in the ODEs by the initial value method, here Rung-Kutta, arriving at the other boundary. Next, a multidimensional root finding problem was solved to find the adjustment of the free parameters at the starting point that zeros the discrepancies at the other boundary point.

At the starting point, zero, there are five starting values $Y_i$ to be specified, but subject to three conditions. Therefore, there are two freely specifiable starting values. Let us imagine that these freely specifiable values are the components of a vector V, which lives in a vector space of dimension two. The components of V might be exactly the values of certain "free" components of Y, with the other components of Y determined by the boundary conditions. Given a particular V, a particular $Y_i(0)$ is thus generated. It can then be turned into a $Y_i(1)$ by integrating the ordinary differential equations to 1 as an initial value problem. Now, at 1, let us define a discrepancy vector Q of dimension two, whose components measure how far these are from satisfying the two boundary conditions at 1. A vector value of V is found, which zeros the vector value of Q. This is iteratively (as many times as required) done by computation in a set of linear equations:

$$[\alpha]\delta V = -Q \tag{18}$$

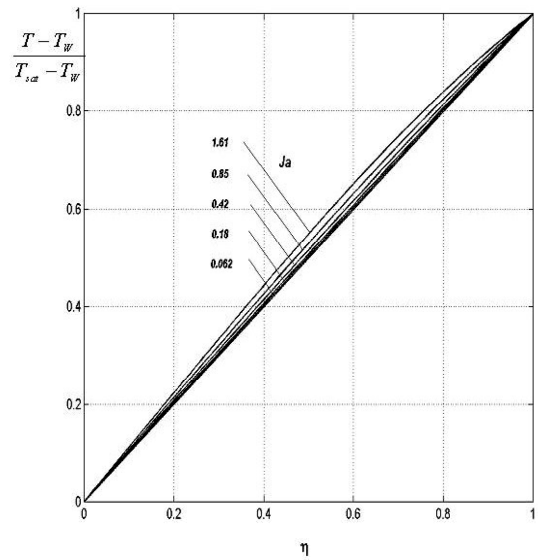


**Figure 2.** Temperature profile for various $Ja$ numbers at Pr=1

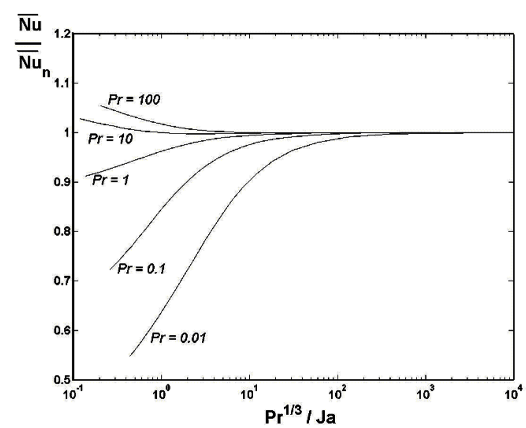


**Figure 3.** Variation of $\frac{\overline{Nu}}{Nu_n}$ versus $(\mathrm{Pr}^{1/3}/Ja)$ for different Pr Numbers

and then adding the correction back,

$$V^{new} = V^{old} + \delta V \tag{19}$$

In (18), the matrix $[\alpha]$ is a two by two matrix and has components given by:

$$[\alpha]_{ij} = \frac{\partial Q_i}{\partial V_j} \tag{20}$$

We might compute these partial derivatives numerically. For this idea, make a small change on $Y_i$ vector and compute vector Q again. Then, there is:

$$\frac{\partial Q_i}{\partial V_j} \approx \frac{Q_i(V_1, V_2 + \Delta V_2) - Q_i(V_1, V_2)}{\Delta V_j} \tag{21}$$

A code has been used to solve the governing O.D.E. equations (12) in such conditions. In this code two boundaries on zero are guessed and then corrected as discussed above. An algorithm of this code is shown in Figure 2.

## RESULTS AND DISCUSSION

### Velocity and Temperature Profiles

From non-dimensional equations, it is clear that for very small Jacob number, the temperature profile is almost linear, as already presented in Nusselt theory [1]. However, as Jacob number increases, the linear approximation of the temperature profile is not true. Figure 2 shows the temperature profile as a function of Jacob number. One can clearly see the linear trend of the temperature profile for low Jacob numbers. In addition, from non-dimensional equations, one can conclude that for viscous fluid, i.e. Pr>>1 the velocity profile is a parabola.

### Heat Transfer Coefficient

Two most important parameters in every heat transfer problem are the heat transfer coefficient and Nusselt number. According to the problem, these two parameters may be defined as:

$$Nu_x = \frac{h_x x}{k} = \frac{-k\left(\frac{\partial T}{\partial y}\right)_{y=o}}{T_{sat}-T_s}\frac{x}{k} = x\left[\frac{\partial\theta}{\partial\eta}\frac{\partial\eta}{\partial y}\right]_{\eta=0} \tag{22}$$

therefore:

$$Nu_x = \left[\frac{gC_p\left(\rho-\rho_v\right)}{4\nu k}x^3\right]^{\frac{1}{4}}\left(\frac{d\theta}{d\eta}\right)_{\eta=0}. \tag{23}$$

The balance between the latent heat transfer and the lateral heat conduction in the film implies that [12,13]:

$$\frac{\rho.h_{fg}.u_c.\delta^2}{k\left(T_{sat}-T_s\right)L} = O\left(1\right) \tag{24}$$

where $u_c$ and $L$ are the reference speed and plate length respectively. Similarly, the balance of the viscous and Archimedean buoyancy forces in the equation of motion require that:

$$\frac{\rho.\nu.u_c}{\delta^2.g.\left(\rho-\rho_v\right)} = O\left(1\right). \tag{25}$$

Using relations above and the rate of heat transfer which is as follows:

$$\dot{Q} \approx \rho.u_c.\delta.h_{fg} \approx \frac{k.\theta_c.L}{\delta}. \tag{26}$$

The order of Nusselt number is:

$$Nu_L = O\left(\frac{\Pr.Ar_L}{Ja}\right)^{\frac{1}{4}} \tag{27}$$

where $Ar_L$ is Archimedes number that is:

$$Ar_L = gL^3\frac{\left(\rho-\rho_v\right)}{\rho\nu^2} \tag{28}$$

finally, the Nusselt number can be written as:

$$Nu_L\left(\frac{4Ja}{Ar_L\Pr}\right)^{\frac{1}{4}} = C_1\left(Ja,\Pr\right). \tag{29}$$

On the other hand, using equation (23), one can write:

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = C_1\left(Ja,\Pr\right).Ja^{-\frac{1}{4}} = C_2\left(Ja,\Pr\right) \tag{30}$$

thus, this parameter has an important rule in explaining the difference of heat transfer rate for high Jacob number with what reported by Nusselt. Note that for $\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = 1$ results tend to Nusselt theory, i.e. a linear temperature profile.

On the other hand, we find some results on plotting the parameter $\frac{\overline{Nu}}{Nu_n}$ versus $\Pr^n.Ja^m$ in various values of m and n, where:

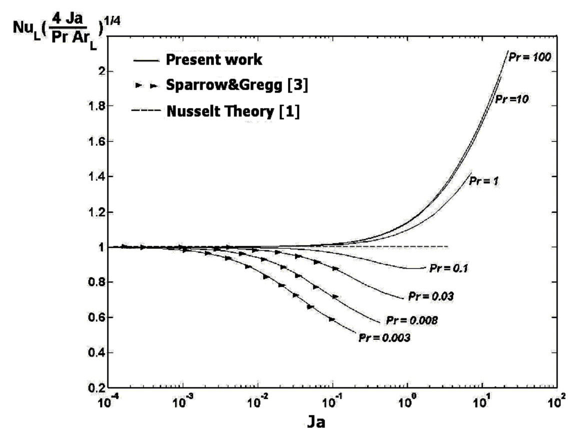


**Figure 4.** Variation of average Nusselt versus Ja number for different Pr numbers

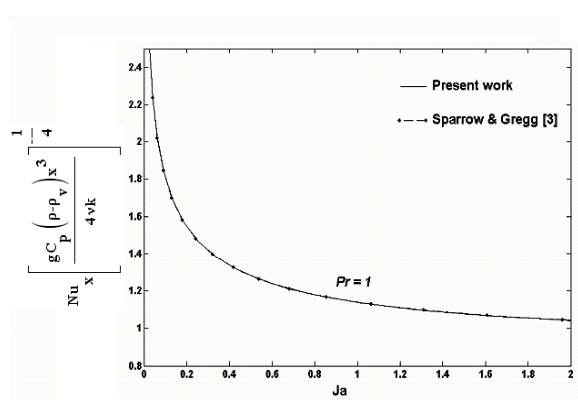


**Figure 5.** Local heat transfer versus Ja number for Pr =1

$$\frac{\overline{Nu}}{\overline{Nu_n}} = \frac{\overline{Nu_L}}{0.943\left[\frac{g(\rho-\rho_v)L^3\acute{h}_{fg}}{\nu k(T_{sat}-T_s)}\right]^{\frac{1}{4}}} = \left(0.68+\frac{1}{Ja}\right)^{\frac{-1}{4}}\left(\frac{d\theta}{d\eta}\right)_{\eta=0} \quad (31)$$

In this relation, $\overline{Nu_n}$ is the original relation presented by Nusselt theory [1]. In Figure 3 this parameter is presented for m=1 and n=1/3, where the power of 1/3 is like forced convection for high Pr. The results for Pr=10 and 100 coincide. Furthermore, we see that for m=1, variation of n has no effect on the results, but for m=-1, by increasing in n from 1/7 to 1, the bottom of lines in plots merge together.

Figure 4 depicted relative Nusselt number for different values of $Ja$ and Pr. It can be seen that the Nusselt theory is acceptable for low Jacob numbers at all Prandtl numbers. The results were also compared with similarity method of Sparrow and Gregg [3] presented for low Pr numbers. When Jacob number increases, the results diverge from the Nusselt theory. These divergences are from level one downward at low Pr and upward for high Pr. One can conclude a resulting profile at Pr=100 which can be used for Pr/Ja<0.1.

A plot of local heat transfer is also given in Figure 5 for Pr=1. The agreement between our results and those of Sparrow and Gregg [3] is excellent. Their calculation technique, utilizing the similarity method, is thus shown to be completely satisfactory for Pr=1.

## CONCLUSION

A computer program has been developed to solve the condensation phenomenon on a vertical plate. A heat transfer coefficient of a condensate film for a wide range of physical properties has been obtained, presented graphically, and discussed. Based on this model the following conclusions can be drawn:

- The Nusselt theory is acceptable for low Jacob numbers at all Prandtl numbers.
- When Jacob number increases, the results diverge from the Nusselt theory.
- These divergences are from level one downward at low Pr and upward for high Pr.
- Results for high Pr (Pr=100) can be used for Pr/Ja<0.1.
- Results for low Pr (Pr=0.01) can be suitable for low Ja (Ja<0.1).

## ACKNOWLEDJMENT

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