JAST, Vol. 2, No. 2, pp 11–18

JAST Journal of Aerospace Science and Technology

Minimum Fuel Trajectory in a 3/D Time Scheduled Climb

S. M. Malaek[1], M. A. Marzaabaadi[2] and S. H. Sadati[3]

A mathematical solution to time-scheduled climb with minimum fuel consumption is presented. The desired final conditions for climb phase are obtained by successive correction of control variables. Aircraft equations of motion are developed based on a point-mass model. The optimality conditions are obtained from Pontryagain's Minimum Principle (PMP). Control variables in the math model consists of the aircraft load factor and its bank angle which both appear in a nonlinear form in the equations of motion, and the throttle setting, which is assumed to vary linearly. Results are computed numerically using Multiple Shooting Method (MSM) and consist of bang-bang control actions for the throttle setting.

NOMENCLATURE

$C(M,h)$	Specific Fuel Consumption
D_0	Zero-lift drag
D_i	Induced drag
$I(n,\eta,\mu)$	Cost function
H	Hamiltonian
M	Mach number
$Q(M,h)$	Fuel Flow Rate Function
$T(M,h)$	Thrust Function
V	Velocity
W_F	Weight of Fuel Consumed
C_{d_0}	Zero-lift Drag Coefficient
h	Altitude
k	Induced-Drag Factor
m	Number of Shooting Nodes
n	Load Factor
q	Dynamic Pressure
W	Weight
x	Range
y	Cross Range
η	Throttle Setting
μ	Bank Angle
χ	Heading Angle
λ	Co-state Variables

1. *Associated Professor, Department of Aerospace Engineering, Sharif University of Technology.*
2. *Aerospace Engineer.MSC.*
3. *Ph.D. Candidate, Department of Aerospace Engineering, Amir Kabir University of Technology.*

INTRODUCTION

With ever increasing demand for air transportation and the existing limitations for the current air routes in addition to the increasing cost of the airport facilities, a new strategy to efficiently use the existing facilities is a must. Comparing to the so-called "Free Flight Concept" one could think of an alternative named as "Fully Scheduled Flight (FSF)". In this approach, a specific amount of time is assigned to each and every phase of the aircraft mission such as climb, cruise or descent. Following this approach, the flight plan carries the usual information regarding the mission as well as a duration tag(DUT) attached to each leg of the mission that shows the time during which the associated mission leg must be carried out. Obviously, DUT must carry some suitable factor of safety based on aircraft type, the geographical characteristics of the region as well as local climatic conditions. As it is known, except for charter and military, all commerical aircraft are subject to a pre advertised schedule for their departure(take-off) and arrivals(landing). However, the current practice only emphasizes on the terminal phases of the flight; that is departure and destination airports. It is therefore, immaterial how the aircraft flies in between. This lack of interest can simply lead to some conflicts among aircrafts interested to fly in a same corridor and it is the intent of FSF to prevent such conflicts during a mission.

To have a complete FSF, one needs to analyze each leg of the flight independently, among which the climb is considered to be the most complicated one due to the changes in the altitude and the resulting effects on the engine performance and drag forces. In this paper a mathematical procedure to solve the problem of time scheduled climb has been presented and for a given aircraft the trajectory that renders the minimum fuel consumption [1] is found for a general climb in the 3/D space. The objective of the mathematical solution is to find velocity, heading angle, flight path angle and altitude in every time step to make sure that the climbing part of the mission is carried out in the given time. Once the trajectory is determined, a specific control system is needed to have the aircraft follow the resulted trajectory.

To make the problem more attractive and considering the fact that different climbing trajectories could be followed in the same given time, a new constraint might be added to the problem. The trajectory of interest here is the one that delivers the least consumption of fuel. The problem of minimum fuel consumption in a vertical plane (2/D) has already been solved analytically [2]. However, in a three-dimensional climb, such as turning climb, no general procedure has yet been developed. On the other hand, to have a minimum fuel climb trajectory, working in a 3/D space as far as practical flights are concerned is necessary. This is due to the fact that a general air routes does not necessarily coincide with the direction of the runway and the aircraft is required to maneuver to get aligned with the air route (Figure (1)).

In this manuscript Pontryagain's Minimum Principle (PMP). [3,4] is applied to find the best trajectory that minimizes fuel consumption for an aircraft in a climbing condition. State variables are velocity, flight path angle, heading angle, altitude, and range, cross range and weight of the fuel consumed during climb. Control variables are load factor, throttle setting (η) and the aircraft bank angle. Selected throttle setting, which is assumed to vary linearly is subjected to the following condition: $\eta_{\min} \leq \eta \leq 1.0$

Obviously, $\eta = 1.0$ shows that throttle is set to its maximum continuous thrust. This parameter could also be examined to find whether the resulted trajectory is a feasible one or not. For example, if a trajectory is found in which $\eta = 1.0$ in all time steps, then this trajectory cannot be considered as an acceptable solution.

EQUATIONS OF MOTION

A 3-D model of the aircraft, assuming all of its mass to be concentrated at its center of gravity has been considered. It is further assumed that all forces and moments are acting at the aircraft C.G position. By these assumptions, the equations of motion can be written as [6,7]:

$$\dot{V} = \frac{g}{W}(\eta T - D_0 - n^2 D_i) - g \sin\gamma \tag{1}$$

$$\dot{\gamma} = \frac{g}{V}(n \cos\mu - \cos\gamma) \tag{2}$$

$$\dot{\chi} = \frac{g}{V\cos\gamma}(n \sin\mu) \tag{3}$$

$$\dot{h} = V \sin\gamma \tag{4}$$

$$\dot{x} = V \cos\gamma \cos\chi \tag{5}$$

$$\dot{y} = V \cos\gamma \sin\chi \tag{6}$$

$$\dot{W}_F = -\eta Q(M,h) \tag{7}$$

$$Q(M,h) = T\max(M,h)C(M,h)$$

While, g is the gravitational acceleration constant and air density $\rho(h)$ and speed of sound $a(h)$ are assumed to have the following relationship [8]:

$$\rho(h) = \rho_0(1 - 6.873E - 6)^{4.26} \tag{8}$$

$$a(h) = 49.02\sqrt{Temp} \tag{9}$$

where, $Temp = 518.67 - .003565h$, then the Mach number is given by $M = V/a(h)$.

Moreover, maximum thrust T, and specific fuel consumption C are assumed to be known functions of Mach number (M) and altitude (h), as follows:

$$T(M,h) = 3(c_1 + c_2 M + c_3 M^2 + c_4 M^3) \tag{10}$$

$$C(M,h) = (d_1 + d_2 M + d_3 M^2 + d_4 M^3) \tag{11}$$

where all constants C_1 through C_4 and d_1 through d_4 are found based on the engine performance curves [12] based on tables1.

It is further assumed that general quadratic drag polar form is applicable, then:

$$D = D_0 + n^2 D_i \tag{12}$$

where:

$$n = \tfrac{L}{W}\ ,\ D_0 = qSC_{d_0}\ ,\ D_i = \tfrac{kW^2}{qS}$$

COST FUNCTION & BOUNDARY CONDITIONS

To find the fuel optimal climb trajectory in its mathematical form, one needs to minimize function I [9, 3], defined as:

$$I(n,\eta,\mu) = -W_F(t_f) \tag{13}$$

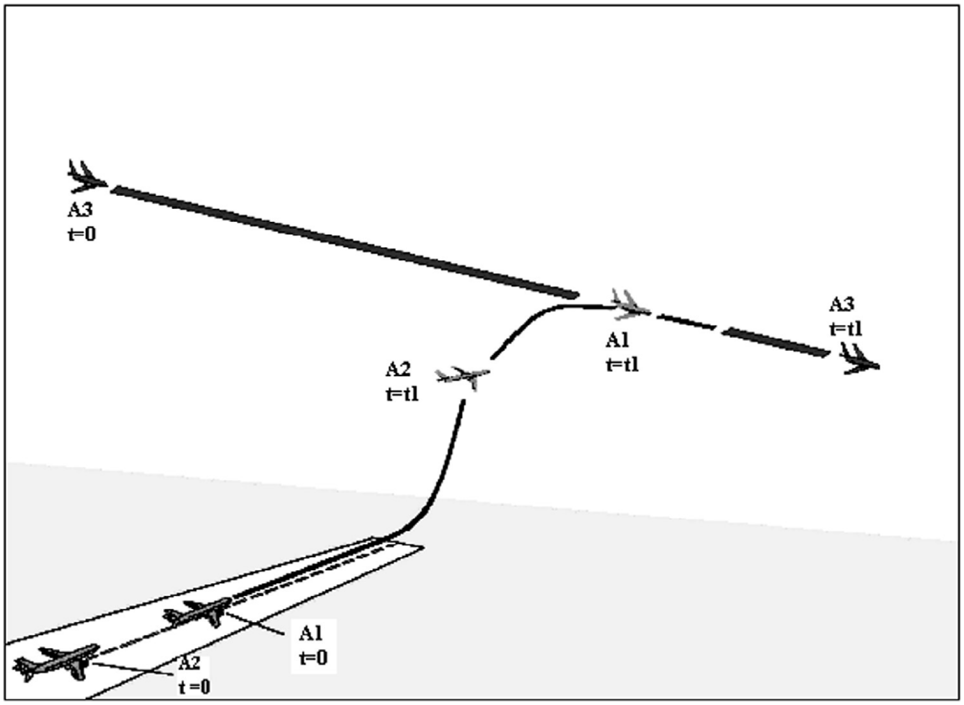

Figure 1. Geometric discribtion of problem

Table 1. Polynomial Coefficients for the Engine Model of Aircraft

Coefficients for Thrust Polynomials T(M,h)

	H^0	h^1	h^2	h^3
C_1	12894.7	-0.27547	-1.18725d-6	6.15751d-11
C_2	-11400.2	0.415365	9.42915d-6	5.09385d-10
C_3	13809.5	-1.72068	7.90316d-5	-9.63618d-10
C_4	-5635.21	1.23973	-7.11898d-5	1.0842d-9

Coefficients for Fuel Flow Polynomials C(M,h)

	H^0	h^1	h^2	h^3
d_1	0.562089	3.79583d-7	1.63423d-10	1.63423d-15
d_2	0.385193	-3.73133d-6	-7.27028d-10	3.76185d-14
d_3	0.063731	1.68353d-6	9.25954d-10	-5.7423d-14
d_4	-0.0299271	-1.04468d-5	8.38085d-11	2.0442d-14

With the boundary conditions given by (14).

$$V(0) = V_0, V(t_f) = V_f$$

$$\gamma(0) = \gamma_0, \gamma(t_f) = \gamma_f$$

$$\chi(0) = \chi_0, \chi(t_f) = \chi_f$$

$$h(0) = h_0, h(t_f) = h_f$$

$$x(0) = x_0$$

$$y(0) = y_0$$

$$W_F(0) = W_{F_0} \tag{14}$$

It is noted that, the final time t_f is a known value, which is defined by the mission programmer. Since, it is desired to climb to a certain condition in the 3/D space with a prescribed duration known as t_f.

OPTIMALITY CONDITIONS

The variational Hamiltonian is formed by adjoining the right-hand sides of the system of differential equations (1) through (7), with the co-state variables or Lagrange multipliers of $\lambda_V, \lambda_\gamma, \lambda_\chi, \lambda_h, \lambda_x, \lambda_y, \lambda_{W_F}$, Then:

$$\begin{aligned} H = {} & \lambda_V(\frac{g}{W}(\eta T - D_0 - n^2 D_i) - g\sin\gamma) \\ & + \lambda_\gamma(\frac{g}{V}(n\cos\mu - \cos\gamma)) \\ & + \lambda_\chi(\frac{g}{V\cos\gamma}(n\sin\mu)) \\ & + \lambda_h(V\sin\gamma) + \lambda_x(V\cos\gamma\cos\chi) \\ & + \lambda_y(V\cos\gamma\sin\chi) + \lambda_{W_F}(-\eta Q(M,h)) \end{aligned} \tag{15}$$

The co-state differential equations are given by:

$$\begin{aligned} \dot{\lambda}_v = {} & -\lambda_V(\frac{g}{W}(\eta T_V - D_{0_V} - n^2 D_{i_V})) \\ & + \lambda_\gamma \frac{g}{V^2}(n\cos\mu - \cos\gamma) + \lambda\chi \frac{gn\sin\mu}{V^2\cos\gamma} \\ & - \lambda_h \sin\gamma - \lambda_x \cos\gamma\cos\chi \\ & - \lambda_y \cos\gamma\sin\chi + \lambda_{W_F}\eta(TC_V + CT_V) \end{aligned} \tag{16}$$

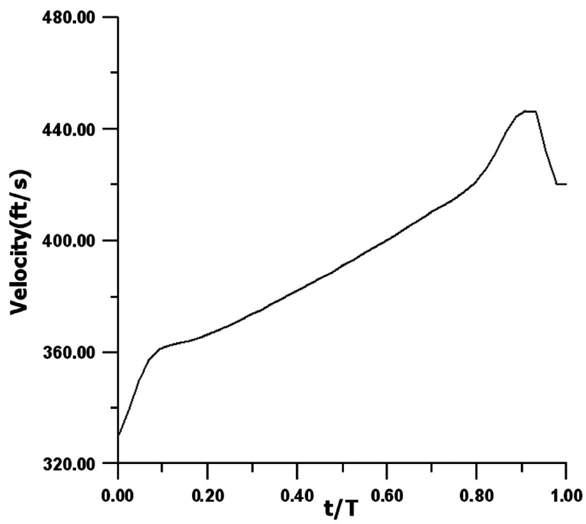

Figure 2. Velocity vs. Scaled Time

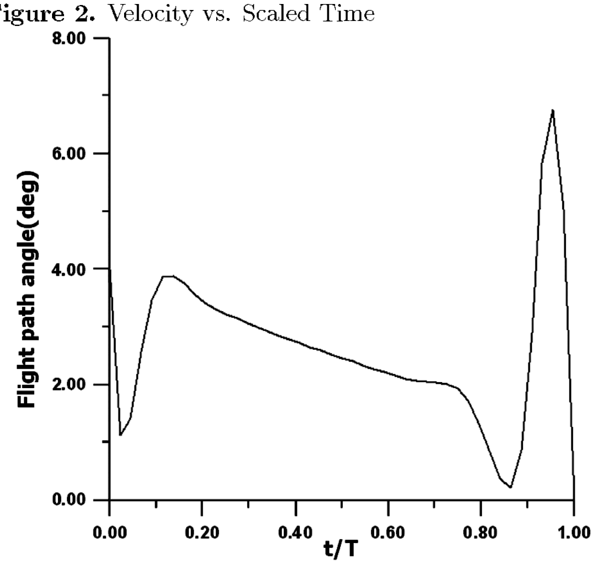

Figure 3. Flight Path Angle vs. Scaled Time

$$\dot{\lambda}_\gamma = \lambda_V g\cos\gamma - \lambda_\gamma \frac{g\sin\gamma}{V} - \lambda_\chi \frac{gn\sin\mu\sin\gamma}{V\cos^2\gamma} - \lambda_h V\cos\gamma + \lambda_x V\sin\gamma\cos\chi + \lambda_y V\sin\gamma\sin\chi \tag{17}$$

$$\dot{\lambda}_\chi = \lambda_x V\cos\gamma\sin\chi - \lambda_y V\cos\gamma\cos\chi \tag{18}$$

$$\dot{\lambda}_h = -\lambda_V\left(\frac{g}{W}(\eta T_h - D_{0_h} - n^2 D_{i_h})\right) + \lambda_{W_F}\eta(TC_h + CT_h) \tag{19}$$

$$\dot{\lambda}_x = 0 \tag{20}$$

$$\dot{\lambda}_y = 0 \tag{21}$$

$$\dot{\lambda}_{W_F} = 0 \tag{22}$$

As can be noticed the problem to find the fuel optimal trajectory is a two-point boundary value problem (TPBVP) with seven states. At the initial point, all seven states are specified and the associated co-states are free to change, whereas, at the final point in time, the values of x, y, W_F are free and the remaining four states are specified. Thus, at the final time all co-states are free except for λ_{W_F} , which is specified by the transversality condition. At the final time, the transversality condition requires that $\lambda_{W_F}(t_f) = -1$ Using the PMP, the control vector U can be determined as:

$$U = \arg[\min_u H(U)] \tag{23a}$$

A necessary condition for unconstrained components of U that satisfy the PMP is:

$$\frac{\partial H}{\partial u} = 0 \tag{23b}$$

Substituting Eq. (15) in Eq. (23-b) with the components of U as n, η and μ yields:

$$\frac{\partial H}{\partial n} = -\lambda_V \frac{2gnD_i}{W} + \lambda_\gamma \frac{g\cos\mu}{V} + \lambda_\chi \frac{g\sin\mu}{V\cos\gamma} = 0 \tag{24}$$

$$\frac{\partial H}{\partial \eta} = \lambda_V \frac{gT}{W} - \lambda_{W_F} CT = 0 \tag{25}$$

$$\frac{\partial H}{\partial \mu} = -\lambda_\gamma \frac{gn\sin\mu}{V} + \lambda_\chi \frac{gn\cos\mu}{V\cos\gamma} = 0 \tag{26}$$

Equation (24) and (25) can be solved simultaneously to obtain the controllers n and μ as:

$$n = \frac{W}{2\lambda_V D_i V}\left(\lambda_\gamma\cos\mu + \frac{\lambda_\chi\sin\mu}{\cos\gamma}\right) \tag{27}$$

$$\mu = \tan^{-1}\left(\frac{\lambda_\chi}{\lambda_\gamma\cos\gamma}\right) \tag{28}$$

However, Eq.(25) does not yield the control η because it appears linearly in the expression for the Hamiltonian (15). In order to determine the control η, the switching function S is defined as:

$$S = \lambda_V \frac{gT}{W} - \lambda_{W_F} CT \tag{29}$$

The control η must be chosen so as to minimize the Hamiltonian (15). However, since, η is a bounded parameter, the throttle control law derived from the PMP (23-a) can be stated as:

$$\text{If } S > 0, \text{ then } \eta = \eta_{\min} \tag{30a}$$

$$\text{If } S < 0, \text{ then } \eta = 1.0$$

$$\text{If } S = 0, \text{ (singular control)} \tag{30b}$$

The generalized Legender-Clebsch [10] condition states that:

$$\frac{\partial^2 H}{\partial n^2} = -\lambda_V \frac{2gD_i}{W} \geq 0 \tag{31}$$

This must be applied to the non-singular control n.

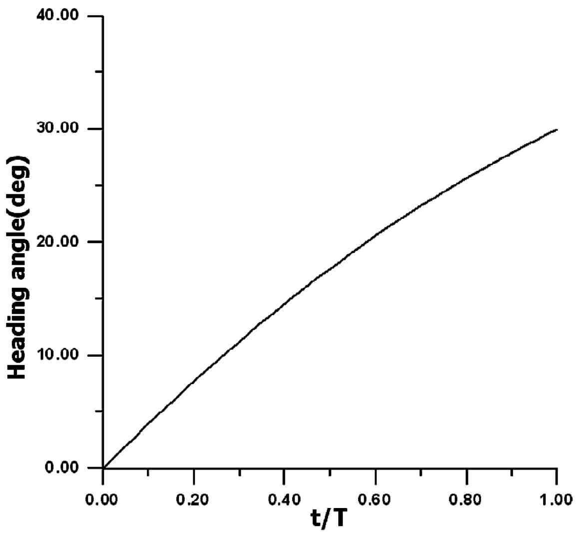

Figure 4. Heading Angle vs. Scaled Time

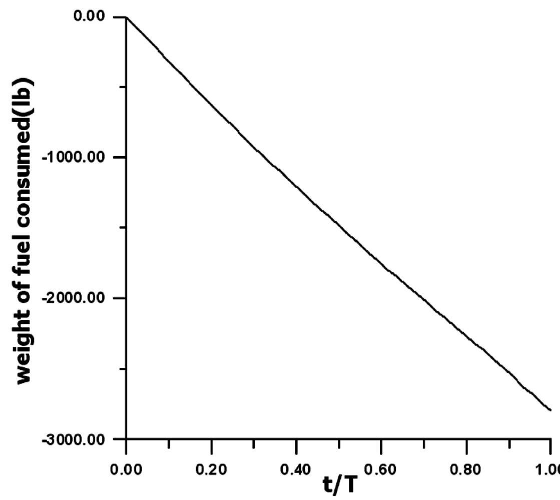

Figure 7. Weight of Fuel Consumed vs. Scaled Time

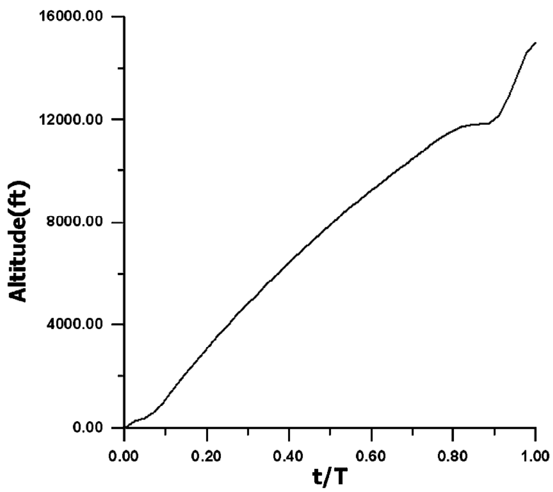

Figure 5. Altitude vs. Scaled Time

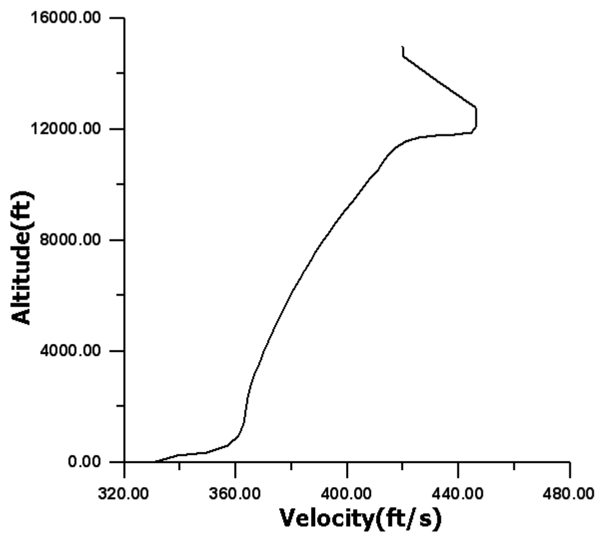

Figure 8. Altitude vs. Velocity

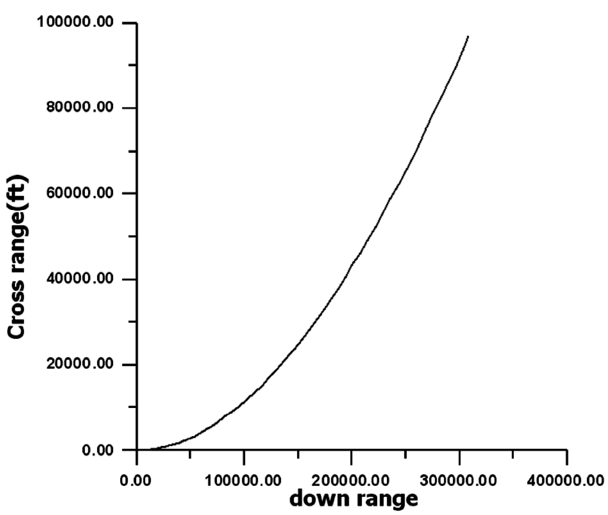

Figure 6. Cross Range vs. Range

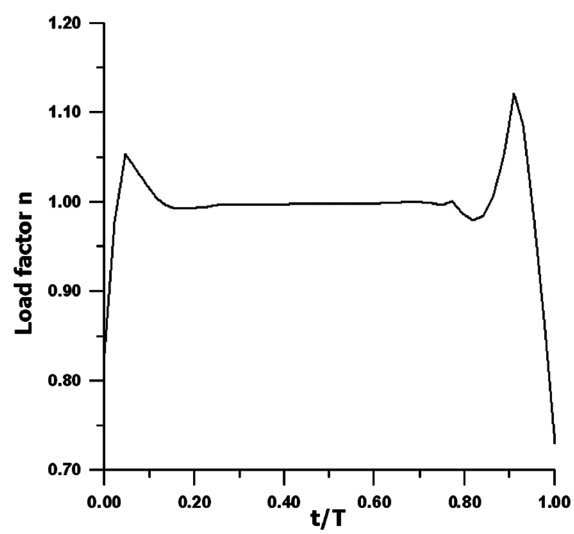

Figure 9. Load Factor Control vs. Scaled Time

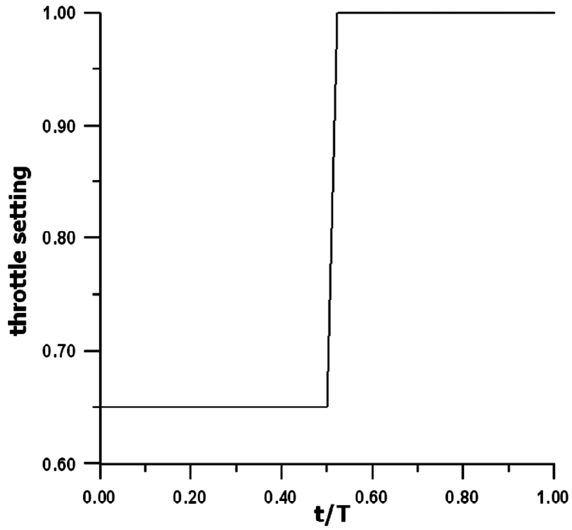

Figure 10. Throttle Setting vs. Scaled Time

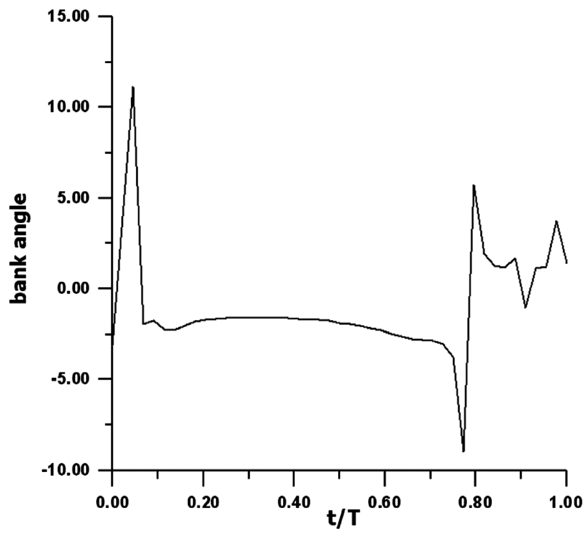

Figure 11. Bank Angle vs. Scaled Time

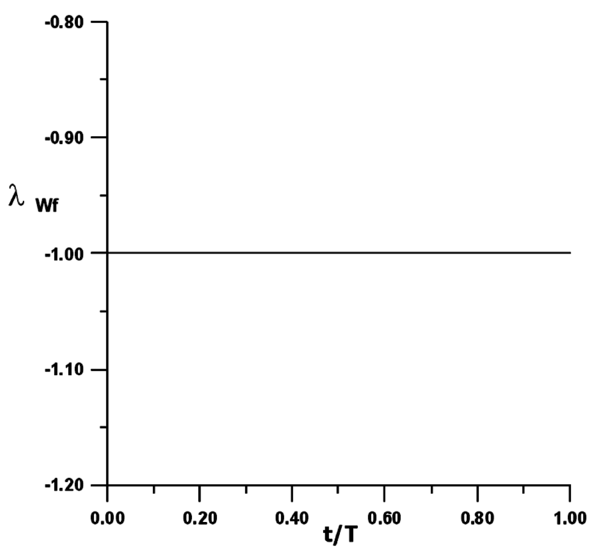

Figure 12. Weight of Fuel Consumed vs. Scaled Time

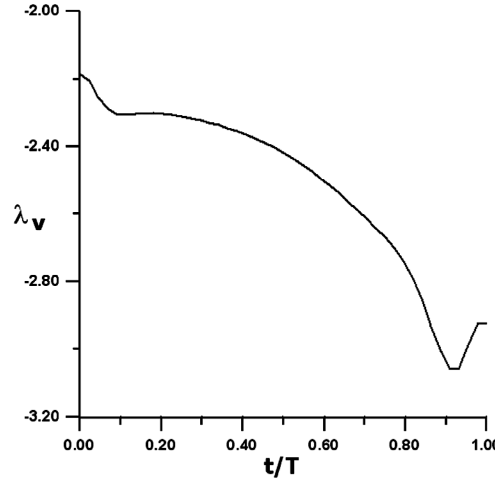

Figure 13. Velocity Adjoin Variable vs. Scaled Time

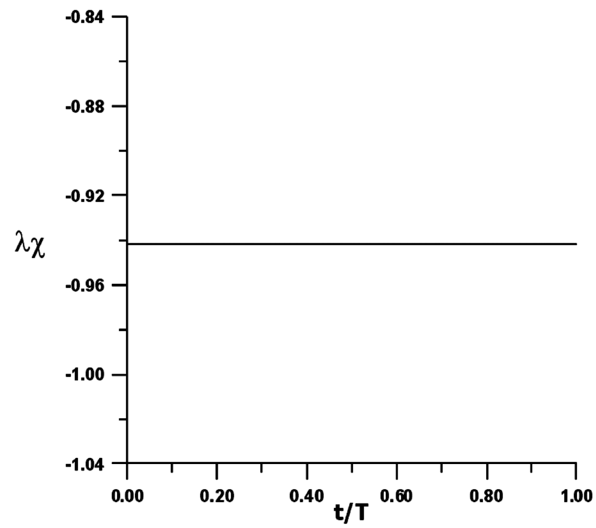

Figure 14. Heading Adjoint variable vs. Scaled Time

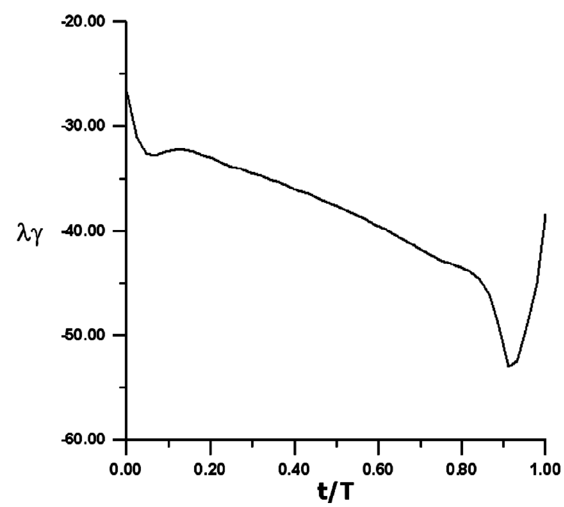

Figure 15. Flight Path Adjoin Variable vs. Scaled Time

CASE STUDY

To show the effectivenes of the current approach, we show how one can find an optimal climb trajectory, for a B-727 that is scheduled to climb from S/L to an air corridor in 15000-ft in 14 minutes. The direction of the air corridor makes a 30-degree angle with the direction of the runway. Therefore, the aircraft is expected to perform a turning climb. To simplify the problem, it is further assumed that the aircraft thrust function varies between 0.65 and the maximum continuous value allowed by the engine manufacturer data. Obviously, the minimum time to reach to the desired altitude of 15000 feet happens while using the maximum continuous thrust (i.e.: η =1.0) on all engines for the whole duration of the climb. This leads to duration of climb equal to 640.0 seconds (10.7 minutes). Since this value is less than 14.0 minutes, then an optimal solution might exist for the specified problem.

As described earlier, applying PMP to the flight-path optimization problem results in a nonlinear TP-BVP where some of the boundary conditions are known at one boundary and some are known at the other boundary. Since, the mathematical formulation is highly non-linear; MSM was used to find the optimal climb trajectory. The basic idea of MSM is to reduce the boundary-value problem at hand to a series of m initial-value problems. The number of initial value problems depends on the nature of the problem at hand. In the case of B-727 in climbing condition one can express the problem as the following: For $j = 1, ..., m-1$, find the numerical solution of the initial-value problems given by:

$$\dot{Z}(t) = \left[F(t, z(t)\right], t_i \leq t \leq t_{j+1} \tag{32}$$

With $z(t_j) = Z_j$. MSM requires fixed subdivisions for the time interval, which has to be chosen by the user, that is:

$$0 = t_1 < t_2 < ... < t_m = t_f$$

initial data for the variables z_i at the time t_j have to be guessed. Assuming the initial guess $Z_j^{(0)}$ for the vector $z(t_j)$; let $z(t; t_j, Z_j)$ denote the solution of the initial-value problem given by (32) in the interval of $[t_j, t_{j+1}]$, then, a trajectory $z(t)$ is a solution to the above multi-point boundary-value problem if and only if the vector $Z = (Z_1, ..., Z_{m-1})^T$ is a zero of $F(Z) = 0$. [3].

Here, the components of $F(z)$ including the continuity or matching conditions are defined as:

$F_j(Z_1, ..., Z_{m-1}) = z(t_{j+1}; t_j, Z_j) - Z_{j+1}, 1 \leq j \leq m-2$

and the boundary conditions are:

$F_{m-1}(Z_1, ..., Z_{m-1}) = [R(Z_1, Z_{m-1})] \in R^n$

with:

$R(Z_1, Z_{m-1}) = [r_i(Z_1, z(t_m; t_{m-1}, Z_{m-1}))]$ and $i = 1, ..., n1 \in R^{n1}$

A zero of the above system of nonlinear equations is determined by a modified Newton method [10,11]. The following features characterize the modified Newton method used here:

1-The Jacobean matrix is approximated via either numerical differentiation or an appropriate Broyden update [10,11].

2-The Newton method is based on a relaxation strategy, where in each iteration the solution of the system of linear equations is via Householder transformations. Taking into account the sparse structure of the coefficient matrix, the latter is just the scaled Jacobean matrix [10,11].

3-The integration method used here for the numerical solution of the initial-value problems is the well-known "Gragg-Bulirsch-Stoer [11] extrapolation method".

Results of fuel optimal climb trajectory for the selected B-727 is presented in Figure.(2) through (8). The adjoin variables $\lambda_{W_F}, \lambda_v, \lambda_\gamma, \lambda_\chi$ which are responsible for the determination of the exteremal controls, are shown in Figs. 12-15. Finally, the controls are shown in Figs. 9-11.

DISCUSSION

This paper present a mathematical procedure to find an optimal climb trajectory during which the fuel consumption is a minimum. All associated optimal controls computed by solving multi point boundary value problems derived based on variational calculus. Since the throttle setting was assumed to follow a linear model, the optimal switching structure had to be found. In the case at hand, this was found to be a bang-bang strategy with one switching point. More investigation shows that the optimal trajectory does not contain singular sub-arcs. Moreover, the resulted bang-bang solution could be shown to be a locally optimal one [4]. More investigations by the authors show that the reduction in fuel consumption in the climb phase is close to %18 which is a considerable saving. The final comment regarding the theoretical results of the optimal theory is the feasibility of commands. Considering the practical limitations dictated by the hardwares involved, one might use suitable filters to choose the most appropriate history for controller commands. Obviously, in such a case the optimal solution for scheduled climb would change to a near optimum solution.

REFERENCES

1. Large, E., "Minimum Fuel Paths for a subsonic Aircraft", *Journal of Aircraft* , **18(5)**, PP 410-414(1981).

2. Burrows, J. W. , "Fuel Optimal Trajectory Computation", *Journal of Aircraft* , **19**, PP 324-329(1982).

3. Bryson, A. E. and Ho, Y.C. , "Applied Optimal Control", Hemisphere, Washington D.C., (1986).

4. Kirk, D. E., "Optimal Control Theory an Introduction", Prentice-Hall Englewood Cliffs. N. J. , (1990).

5. Sadati, S. H, "Time-Optimal Reorientation Maneuvers of an Aircraft", Amir- kabir University, Iran, (1998).

6. Marzaabaadi, M. A., "Climbing Trajectory in Scheduled Flight", Sharif University of Technology, (1996).

7. Bilimoria, K. D and Cliff, E. M., "Singular Trajectory in Airplane Cruise-Dash Optimization", *Journal of Guidance, Control and Dynamics*, **12**(3), PP 304-310(1989).

8. Bertin, J. J. and Smith, M. L. , "Aerodynamics for Engineers", Prentice-Hall, (1989).

9. Oberle, H. J., "Numerical Computation of Singular Control Functions in Trajectory Optimization Problems", *Journal of Guidance and Dynamics* , **13**(1), PP 153-159(1990).

10. Press, Teukolsky and Flannery , "Numerical Recipies in FORTRAN", (1994).

11. Stoer J. and Burlisch R. , "Introduction to Numerical Analysis", Springer Verlag, (1992).

12. Corning, G. , "Supersonic & Subsonic CTOL & VTOL Airplane Design", College Park Maryland, (1979).