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Scientific-Research Article

WENO families in accuracy and computational costs

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In this paper a comparison of Weighted Essentially Non-Oscillatory (WENO) scheme is presented and different kinds are compared. High resolution schemes are one of the best ways decreasing the cost of processes and also increasing the resolution as is clear. Different WENO's influence on the weights that applies on the neighborhood of the cells that is supposing to be calculated. Mentioned schemes, were tested on wave equation at first and in continued with the first and second dimension test cases. 3rd ,5th, 7th and 9th order of JS-WENO, MWENO, ZWENO and MZWENO are compared in Goethe tests. This scheme was applied in finite volume characteristic wise algorithm in order to reach much more accuracy. Buckley-Leverette, Sod shock tube, Shu-Osher, Lax test, Riley-Taylor instability and double Mach reflection test cases was compared. As the result, MZWENO in equal order with the other ones would report more accurate reply. But as a new research here we showed that e.g. although MZWENO 5th order could promote the accuracy of the scheme up to about two times higher, but the cost of computing will increase more than the JS 7th order one. So, it is concluded that employing 7th order of JSWENO leads to higher accuracy with less computational costs.

Keywords: *WENO, Characteristic, finite volume, Lax Friedrich flux splitting, computational costs*

Introduction

Hyperbolic conservation laws arise in a wide range of applications in science and engineering, such as aerodynamics, meteorology and weather prediction, astrophysical modeling, multi-phase flow problems, and the study of explosion and blast waves.[1] Numerical stability is one of the problematic criteria for solving flow equations that would be decreased in increasing of resolution. First order methods will not predict exact results as they include numerical dissipation. In the other hand if high resolution one's does not have limiters they would suffer from nonphysical oscillations in high gradient situations. In last three decades, plenty of high resolution schemes was presented and developed to capture discontinuities such as shock waves based on Riemann solution in finite volume and finite difference discretization. These methods rarely were extended in finite element. Another gist in numerical methods is interpolating and extrapolating of variables which are not solved; that may UPWIND be the simplest way which has high dissipation. HYBRID and POWERLOW methods are created to diminishing these dissipations [2], [3]. Appliance of these limiters on high resolution schemes leads to reach high accuracy and also reliable answers in sophisticated phenomenon. Some limiters are such as: Total Variation Diminishing (TVD), NVD, NVF and also BVD has been presented. Negative density

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and pressure are the problems that may be interfaced with in solving equation process. First solution that is considered to be nice is the replacement of these nonphysical negatives with the positive ones. But these replacements are nor conservative and a logical solution for stability. So, solution method should be intrinsically positive and in order to satisfy this condition positivity preserving methods were presented such as: Godunov-Type[4], flux vector splitting[5], Lax-Friedrich flux splitting [6], [7], HLLC[8], Gas kinetic schemes [9], [10] and recently Yan Guo presented a new WENO finite volume positivity preserving scheme[11]. Accuracy, stability and low computing process are the criterion of worth numerical methods. Accuracy generally will increase by further ways: Employing more node in meshes, high order interpolating formulas control on some characteristics of the mesh like queuing. But applying each of them leads to high computational costs. Although there are a lot of attempts to improve these brainstorming's. In dealing with severe gradients to preventing oscillations in high order methods some new technics were introduced that are grouping in two categories: Artificial viscosity methods, Total variation diminishing methods, That the second option is the selected one we have chosen in this paper. As the former works of each research is the basic principle of the study such these papers are mentioned here. Unlike TVD and NVD schemes that change to first order interpolating in sever gradients, ENO and WENO schemes use high order ones. WENO has been interested for researchers in recent decades that was first introduced by Liu[12]. Instead of choosing one stencil as in ENO do, WENO use all proportional one's in order to achieve higher accuracies. The resulting weighted ENO schemes based on cell averages and a TVD RungeKutta time discretization [12]. Levy et al based on a centered version of the WENO presented a family of high-order ENO, central schemes for approximating solutions of hyperbolic systems of conservation laws [13]. As JS-WENO scheme may not always be monotonicity preserving but coupled with the monotonicity preserving bounds of Suresh (1997) they perform very well. Balsara et al. presented a newer version. They introduced MPWENO that the resulting monotonicity preserving weighted essentially nonoscillatory schemes had high phase accuracy and high order of accuracy. Presented scheme were also efficient and do not have a computational complexity that is substantially greater than that of the lower-order members of this same family of schemes[14]. Henrick et al. developed a fifth-order WENO scheme. Necessary and sufficient conditions on the weights for fifth-order convergence are derived; one more condition than previously published is found. They founded magnitude of a parameter which keeps the weights bounded. A simple modification of the original scheme were founded to be sufficient to give optimal order convergence even near critical points[15]. Huankun Fu added WENO to WCS so that the scheme has appropriate dissipation and to eliminate oscillations; and also instead of using a black box subroutine for derivative, they only calculate the weights for pressure and density outside the subroutine at each time step[16], [17]. Yan Guo et al by combining lower order compact stencils with WENO nonlinear weights to reach a higher order finite volume compact-WENO scheme[11]. Gerolymos et al. tabulated WENO coefficients up to r=9 (WENO17) [18]. Improved WENO schemes originally have been designed to capture sharper discontinuities than the classical fifth order Jiang–Shu scheme does, were evaluated for the purpose of implicit large eddy simulation of free shear flows that was researched by Zhao et al [19]. Tao et al. proposed a central finite volume framework that involves Hermite WENO. Compared with central WENO methods, the spatial reconstruction used there is much more compact; and unlike the original HWENO methods, their proposed schemes require neither flux splitting nor the use of numerical flux [1]. Mullenix et al. developed a fifth order scheme that maintains the design order of accuracy of the underlying scheme while maximizing the bandwidth resolving efficiency[20]. Sun et al. used characteristic WENO forms for computing inviscid fluxes in direct numerical simulations of compressible turbulent flow over wavy wall geometries that have been carried out by solving N–S equations on general curvilinear coordinates [21]. San et al. presented several joint solvers that were developed within the framework of the reconstruction and flux-splitting approaches using the underlying MUSCL and WENO frameworks. The modular development of these joint solvers provides an ease in characterizing the solution procedures. Demonstration of the reconstruction based WENO scheme with Roe solver was more accurate than all the versions of the flux-splitting WENO solvers tested in that study was illustrated. Proving that the results are highly dependent on the choice of the flux limiter and the tests on capturing Kelvin–Helmholtz instability were illustrate there. [22]. In this paper, implementation of WENO in characteristic and conservative variable forms are going to be explained. Although the way that the scheme should be applied to these variables are available but the comparisons and detail of each application and computational costs is not available within a paper to compare. Here, characteristic variables are employed to solve Euler equations and different WENO types with their computational costs are illustrated to demonstrate whether these new cousins are noteworthy in wrapped cases or not.

Algorithm

Characteristic variables which are based on the conservative differences are explained to be enlisted in WENO scheme [23]. Characteristic Algorithm For all (i+1/2): Right and Left eigenvectors and eigenvalues. For all (i+1/2): compute divided differences of fluxes (ΔF) and conservative variables (ΔU) coincident order of accuracy e.g. for 5th order five divided differences are needed.

$$\begin{aligned} \Delta F_{i+1/2} &= F_{i+1} - F_i \\ \Delta U_{i+1/2} &= U_{i+1} - U_i \end{aligned} \tag{1}$$

In each (i+1/2): depending on the flux splitting, and also requirements of both the left and right running flows the left and right cell face of the fluxes should be computed, here Lax-Friedrich Flux splitting were used,

$$\begin{aligned} \Delta F^{+}_{i+1/2} &= \frac{1}{2}(\Delta F + \lambda \Delta U) \\ \Delta F^{-}_{i+1/2} &= \frac{1}{2}(\Delta F - \lambda \Delta U) \end{aligned} \tag{2}$$

In accordance with the cell faces calculating left ($W-$) and right characteristic variables ($W+$) of the cells,

$$\begin{aligned} W^{+}_{i+5/2} &= L_{i+1/2} \times \Delta F^{+}_{i+5/2} \\ W^{+}_{i+3/2} &= L_{i+1/2} \times \Delta F^{+}_{i+3/2} \\ W^{+}_{i+1/2} &= L_{i+1/2} \times \Delta F^{+}_{i+1/2} \\ W^{+}_{i-1/2} &= L_{i+1/2} \times \Delta F^{+}_{i-1/2} \\ W^{-}_{i-3/2} &= L_{i+1/2} \times \Delta F^{-}_{i-3/2} \\ W^{-}_{i-1/2} &= L_{i+1/2} \times \Delta F^{-}_{i-1/2} \\ W^{-}_{i+1/2} &= L_{i+1/2} \times \Delta F^{-}_{i+1/2} \\ W^{-}_{i+3/2} &= L_{i+1/2} \times \Delta F^{-}_{i+3/2} \end{aligned} \tag{3}$$

Use WENO to compute the corresponding fluxes of characteristic variables and transform back them into the conservative form. Solve Euler equation by using cell faces which was earned from WENO. Now but haw should WENO be implied into characteristic forms? In the smoothness indicator equation forms, there is an expression that acknowledging differences. This is the main distinct part between employing conservative and characteristic WENO. In fact, the differences on i+1/2 could be simply define as [(i+1)-(i)]. So, the corresponding characteristic variable could be alternate with the expression of distinct in smoothness indicators and polynomials proportionate. In the following paragraph the formulas for the 5th order accuracy is given. Allocate each characteristic variable appropriate with the differences of conservative one's and construct smoothness indicators ($IS\pm$).

$$IS^{+}_{1}=(\tfrac{1}{3})\times\left\{W^{+}_{i+\frac{5}{2}}(4W^{+}_{i+\frac{5}{2}}-11W^{+}_{i+\frac{3}{2}})+10\left[W^{+}_{i+\frac{3}{2}}\right]^{2}\right\}$$
$$IS^{+}_{2}=(\tfrac{1}{3})\times\left\{W^{+}_{i+\frac{3}{2}}(4W^{+}_{i+\frac{3}{2}}-5W^{+}_{i+\frac{1}{2}})+4\left[W^{+}_{i+\frac{1}{2}}\right]^{2}\right\}$$
$$IS^{+}_{3}=(\tfrac{1}{3})\times\left\{W^{+}_{i+\frac{1}{2}}(10W^{+}_{i+\frac{1}{2}}-11W^{+}_{i-\frac{1}{2}})+4\left[W^{+}_{i-\frac{1}{2}}\right]^{2}\right\} \quad (4)$$
$$IS^{-}_{1}=(\tfrac{1}{3})\times\left\{W^{-}_{i-\frac{3}{2}}(4W^{-}_{i-\frac{3}{2}}-11W^{-}_{i-\frac{1}{2}})+10\left[W^{-}_{i-\frac{1}{2}}\right]^{2}\right\}$$
$$IS^{-}_{2}=(\tfrac{1}{3})\times\left\{W^{-}_{i-\frac{1}{2}}(4W^{-}_{i-\frac{1}{2}}-5W^{-}_{i+\frac{1}{2}})+4\left[W^{-}_{i+\frac{1}{2}}\right]^{2}\right\}$$
$$IS^{-}_{3}=(\tfrac{1}{3})\times\left\{W^{-}_{i+\frac{1}{2}}(10W^{-}_{i+\frac{1}{2}}-11W^{-}_{i+\frac{3}{2}})+4\left[W^{-}_{i+\frac{3}{2}}\right]^{2}\right\}$$

Alpha (αr ±) and weights (ωr ±) should be calculated by the matching (dr ±) coefficients.

$$\alpha^{\pm}_{r}=\frac{d^{\pm}_{r}}{(\varepsilon+IS^{\pm}_{r})^{2}} \qquad \omega^{\pm}_{r}=\frac{\alpha^{\pm}_{r}}{\sum_{j=0}^{k-1}\alpha_{j}} \quad (5)$$

These weights should be multiplied to proportional polynomials. Because of simplicity of WENO, after Liu et al brought up this scheme in 1994 a lot efforts have been done to improve this kind of scheme. Although these schemes are all based on the former scheme that was mentioned in the last section, but there are some changes specially in calculating alphas. In the other hand some other publications all over the world combined this scheme with other schemes that leads to higher order of accuracies [11], [24], [25], [13], [26], [20], [27] and [16] but here we are not going to discuss around them. Henceforth, we call the mentioned scheme as M-WENO , Z-WENO and other kinds formulation such as M-WENO, Z-WENO and the combination of these two's, MZ-WENO is considered forward. M-WENO:

$$\omega^{M}_{r}=\frac{a^{M}_{r}}{\Sigma^{2}_{j=0}a^{M}_{j}} \qquad a^{M}_{r}=g_{r}(\omega^{JS}_{r}) \qquad gr(\omega)=\frac{\omega(d_{r}+d^{2}_{r}-3d_{r}\omega+\omega^{2})}{d^{2}_{r}+\omega(1-2d_{r})} \quad (6)$$

Z-WENO:

$$\omega^{z}_{r}=\frac{\alpha^{z}_{r}}{\Sigma^{2}_{j=0}\alpha^{z}_{j}} \qquad \alpha^{z}_{r}=\frac{d_{r}}{IS^{z}_{r}} \qquad \frac{1}{IS^{z}_{r}}=1+\left(\frac{t_{5}}{\varepsilon+IS^{JS}_{r}}\right)^{2};\ t_{5}=\left|IS^{JS}_{0}-IS^{JS}_{2}\right| \quad (7)$$

MZ-WENO:

$$\omega^{MZ}_{r}=\frac{\alpha^{MZ}_{r}}{\Sigma^{2}_{j=0}\alpha^{MZ}_{r}} \qquad a^{M}_{r}=g_{r}(\omega^{Z}_{r});r=0,1,2 \quad (8)$$

Results

Test cases are a method to distinguish how the schemes are accurate and the CPU time that each one takes long. 1D and 2D inviscid test cases are presented in this literature. Euler equation in form of characteristic variable was employed for solving the algorithm of these tests.

Sod shock Tube

Sod shock tube was tested with the conditions that is explained further[28]. Where $x \in (0.1)$ and the responds were compared at time t=0.25. Fig. 1

$$(\rho,u,p)=\begin{cases}(1,0,1) & t=0,x\le 0.5;\\ (0.125,0,0.1) & t=0,x>0.5;\end{cases} \quad (9)$$

Shu-Osher Problem

Shu-Osher is another 1-dimension test case to comparison of different schemes [29]. The initial condition for this Riemann problem is mentioned bellow. Where $x \in (-5{\sim}5)$ that in Fig. 2 density plot is zoomed at the neighborhood of 1.5 for more obvious shown.

$$(\rho,u,p)=\begin{cases}(3.85,2.629,10.3) & t=0,x\le -4;\\ (1+0.2Sin(5x),0,1) & t=0,x>-4;\end{cases} \quad \textbf{(10)}$$

Lax Shock Tube

Lax shock tube [30] is another test case that is going to be examined to distinguish whether the most accurate scheme is like the former ones or not, initial conditions are mentioned further then the consequent results are shown in Fig. 3.

$$(\rho,u,p)=\begin{cases}(0.445,0.698,3.528) t=0,x<0;\\ (0.500,0.000,0.571) t=0,x\ge 0;\end{cases} \quad \textbf{(11)}$$

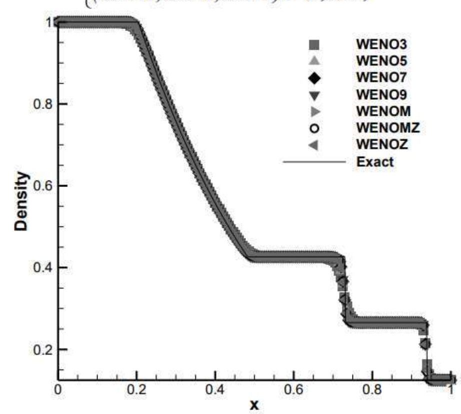

Fig.1a: Density plot of sod shock tube

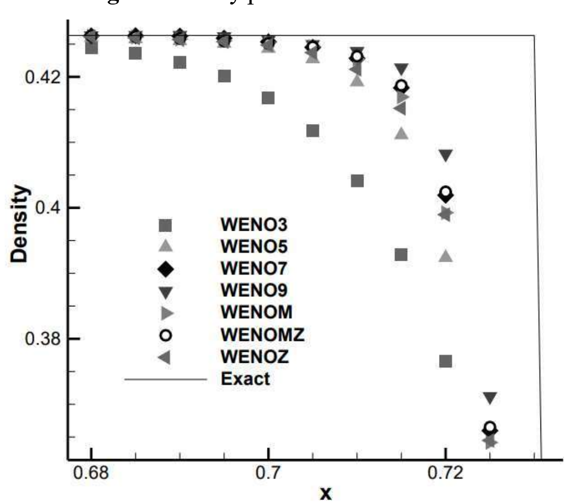

Fig. 1b: Zoomed density plot of sod shock tube

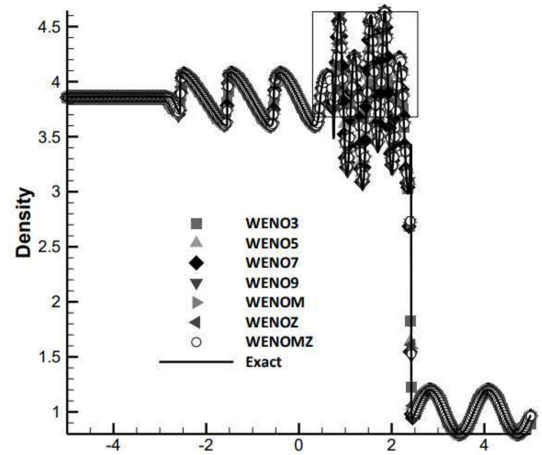

Fig.2a: Density lines of Shu-Osher problem

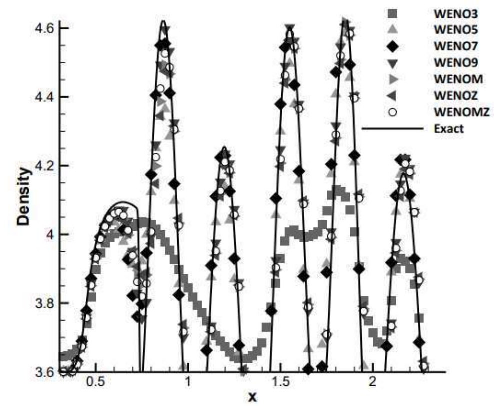

Fig. 2b: Zoomed density lines of Shu-Osher problem

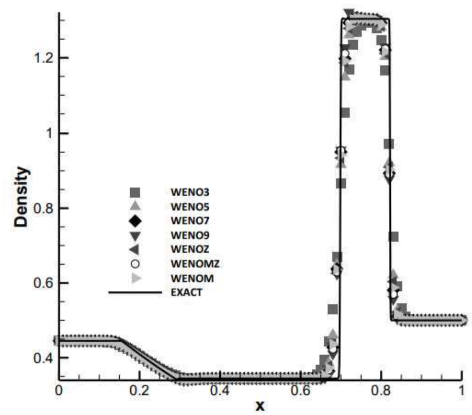

Fig. 3a: density of Lax Shock tube

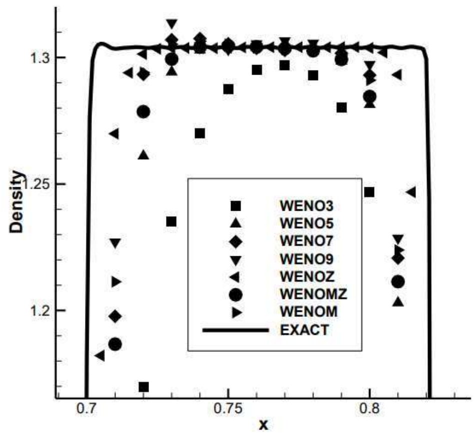

Fig.3b: Zoomed density lines of Lax Shock tube

Rayleigh-Taylor instability

Rayleigh-Taylor instability details that is defined at [31] is compared as a two-dimension test case that is a kind is validation to our code. Different schemes are obviously distinguishable for our goal. [120×480]

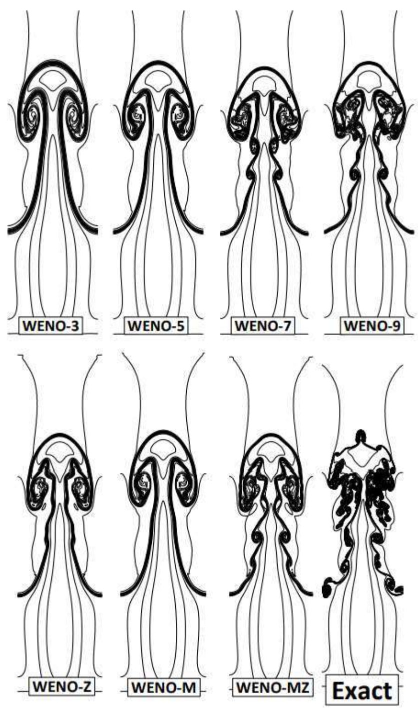

Fig. 4: density contour of Rayleigh-Taylor instability

resolution at time t=1.95 with CFL=0.5 and γ=5/3 is tested here. Fig. 4 is the results illustration.

$$(\rho,u,v,p)=\begin{cases}(2,0,\frac{-1}{4}\sqrt{\frac{\gamma p}{\rho}}\cos(0.8\pi\,\mathrm{x}),2\,\mathrm{y}+1);x\le 0.5;\\(1,0,\frac{-1}{4}\sqrt{\frac{\gamma p}{\rho}}\cos(0.8\pi\,\mathrm{x}),y+\frac{3}{2});x>0.5;\end{cases}\quad(12)$$

In this case as the comparison benchmark for the exact one [360×1440] resolution by WENO-9 has been handled. Form Fig.4 it could be concluded that MZ is able to capture more details of turbulences than JS7.

Lax Configuration

Two LAX configurations were employed that the initial conditions are mentioned at [32]. Configuration number 4, 6, 8 and 17 that are compared has the grid [400×400] that the results where compared at time t=0.3(s) and CFL number were adopted equal to 0.2499 Fig. 5 and Fig. 6 are monitoring Lax Con number 8 and 17 respectively. For the exact solution, grid [1200×1200] were emplyed as a comparison benchmark.

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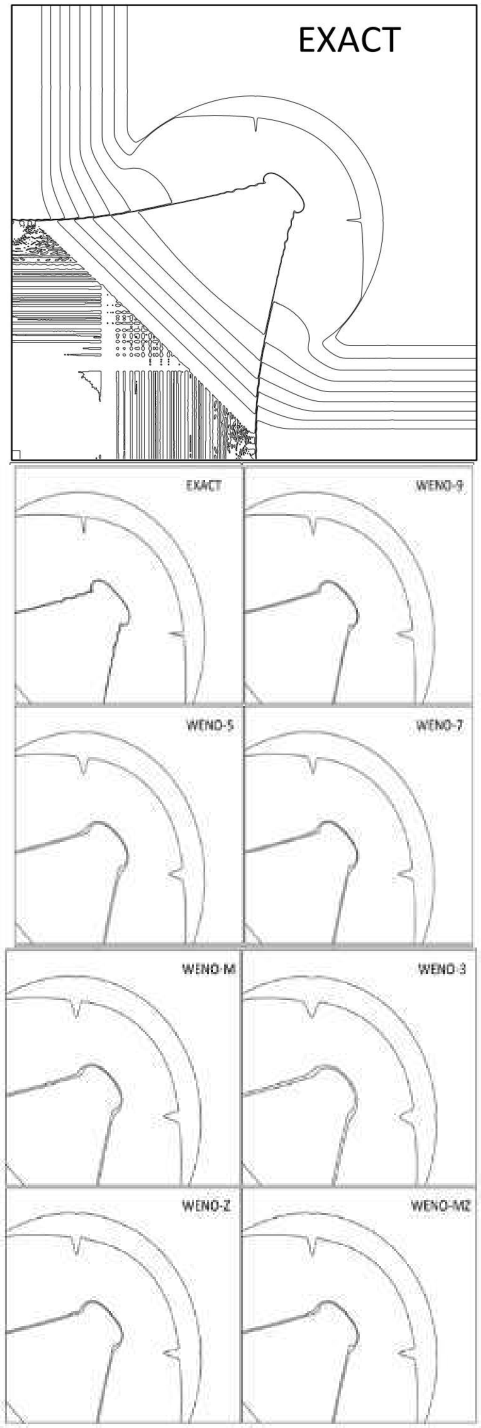

Fig. 5: Density contour of Lax number 8

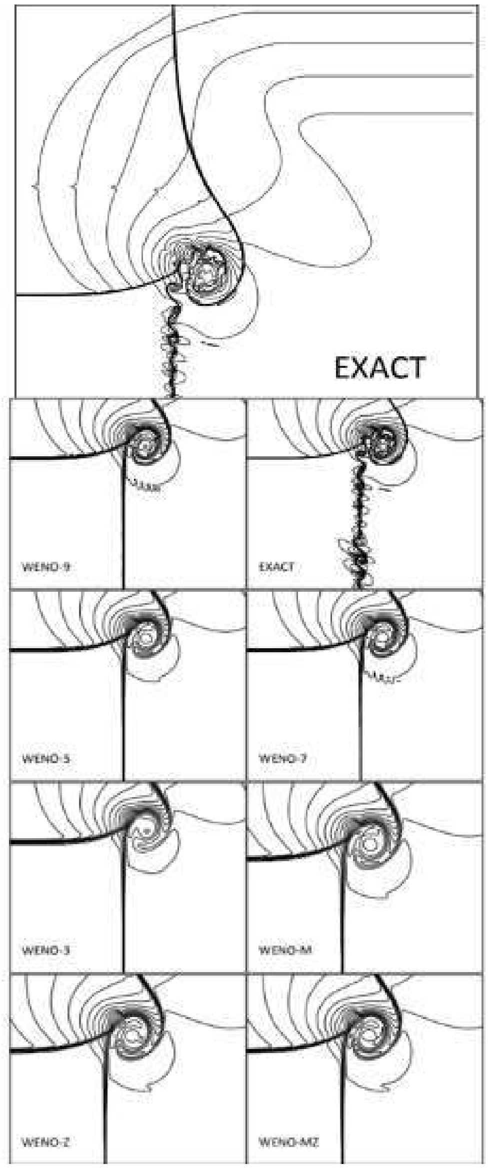

Fig. 6: Density contour of Lax number 17.

As illustrated in former pictures there is not a significant difference between those schemes that are mentioned, albeit a bit higher accuracy could be extracted from Fig.6 for the 7th one.

Double Mach Reflection

Double Mach Reflection test case were first analyzed by H. Li [33]. The resolution of the mesh was adopted as [960×240] that the result was compared at t=0.2 (s). proportional results are shown in Fig. 7 and the results of those two schemes are somewhat similar to each other.

$$(\rho,u,v,p)=\begin{cases}(\gamma,0,0,1)\ ;(x-1)^2+(y-0.5)\leq 1\\(8,7.144,-4.125,116.5);else\end{cases} \quad (13)$$

Conclusion

The main purpose of the paper is the comparison of various WENO families for the sake recognize the most affordable scheme. Fig. 8 is the computational cost of those schemes proportional to WENO-3. For more detail as the MZ one can improve the accuracy up to about two times better (5th order to about 7th) the research has focused on these two's. As it can be seen in Fig. 8 the CPU time of MZ and 7th order is almost equal, however, in most cases it is a bit more. Therefore, from the former results (Fig. 1 to Fig. 7) it is obvious that in most cases whether the 7th order one is more accurate than the MZ one or in some cases they are at least as sharp as each other. Accordingly, in general selecting the 7th order one as the algorithm scheme seems more logical than the MZ one and also could be generalize for the other orders.

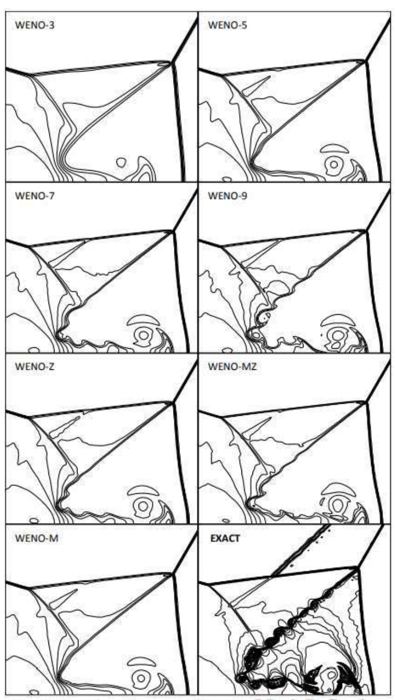

Fig. 7: Density Contour of Double Mach Reflection

Table 1: CPU time of WENOJS-3

Test Case	WENOJS-3 CPU Time (second)
Sod-Shock Tube	0.468
Lax-Shock Tube	0.1092
Shu-Osher	2.4492
Reighly-Tailor	155.2366
LAX-8	81.29212
LAX-17	111.3535
Double Mach Reflection	429.4

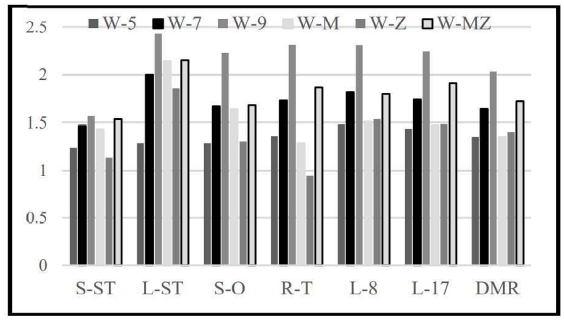

Fig. 8: Proportional CPU Time

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