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A Parametric Study for Vibration Analysis of Composite Cylindrical Shell Resting under Elastic Foundation: Analytical and Numerical Methods

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The aim of this study is to investigatethe effective parameters on vibrations of circular cylindrical shells with fixed rotary speed andresting elastic foundation by means of analytical and finite element numerical simulation. First, the governing equations are derived using the theory of Donnell, considering the centrifugal forces,Coriolis acceleration, and the initial annular tension. Then, the analytical solution for cylindrical shells isintroduced under simply supported conditions. Further, the effect of parameters such as therotational speed of the shell, its lay-up, fiber angle, and the stiffness of the elastic foundation on the values of natural frequency and the critical velocity of the shells are studied. The analytical solution results are in good compatibility with the results achieved from the finite element method.

Keywords: Vibration analysis, Annular cylindrical shells, Donnel theory, Analytical solution, Finite element

Introduction

A lot of studies have been done on the free vibration of cylindrical shells. Egle and Sewall [1] have studied the effect of stiffeners on the variations of natural frequencies of the shells reinforced with rings and stringer, under different boundary conditions. Stiffeners are treated asdiscrete elements, and Hamilton's principle and the energy methods have been applied so as to obtain the equations of motion. Reinhart and Wong [2] have investigated the vibrations of the shells reinforced with stringer, which are considered asdiscrete elements. Amabili [3] studied the large amplitude (geometrically non-linear)vibrations of annular cylindrical shells, under different boundary conditions, exposed to harmonic radial excitationin both empty and full of fluid states. Karagiozisaet al. [4] investigated the nonlinear vibrations of circular cylindrical shellsin two states of empty and full of fluid,with cantilever conditions, exposed to harmonic radial excitation force. Pellicano[5] studied linear and nonlinear vibration analysis of circular cylindrical shells under various boundary conditions. He used the Sanders-Kwittertheory, and extended the displacement fields as combination of double series ofharmonic functions.

Shaoand Ma [6] analyzed the layered circular cylindrical shells under arbitrary boundary conditions using Fourier series expansion. Eipakchi et al. [7] extracted the homogeneous and isotropic equations for variable wallby means of*FSDT* and later solved them using the perturbation theory. Zhao and Liew [8] investigated the vibrations of composite annular cylindrical shells reinforced with orthogonal stiffeners. Jafariand Bagheri [9] investigated the free vibration analysis of isotropic cylindrical

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shells with numerical, analytical, and laboratory methods, and then, compared the results of these methods with each other. Ebrahimi and Najafizadeh [10] investigated the free vibration analysis of the FGM cylindrical shell in the two-dimensional state, in 2014. The equations of motion were derived according to the LoveLaw and the classical theory. In this study, a comparison analysis of the conventional results available in the literature and the two-dimensional mode results wasconducted. In 2014, Jin et al. [11] investigated the analysis of three-dimensional free vibration of thick cylindrical shell based on elastic foundation. In their study, the Riley-Ritz method was used to solve the equations.In this paper, the study of vibrations of composite annular cylindrical shells with fixed rotary speedwhile positioned on the elastic foundation was conductedusing analytical and numerical methods. The governing equations were derived by means of Donnel theory. Then, the effect of various parameters on the frequency of the structure was discussed. Later, the results of analytical and numerical methods were compared with each other.

Governing equations

In Fig. 1, the geometry and coordinate system of a cylindrical shellwith length l, mean radius R, and thickness h are depicted. The origin o is located atthe top end and inthe middle plane of the shell. x, y, and z are in the axial, circumferential, and outer normal directions, respectively.

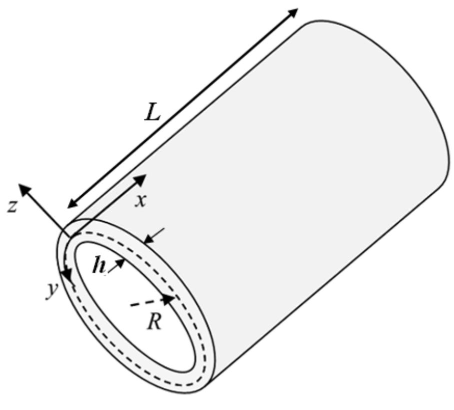


Figure 1. Geometry and coordinate system of a cylindrical shell

Components of the displacement vector of the shell are stated herein, based on the classical model and Donnell theory, as follows[12]:

$$u_1(x,y,z,t)=u(x,y,t)-z\frac{\partial w(x,y,t)}{\partial x},$$
$$u_2(x,y,z,t)=v(x,y,t)-z\frac{\partial w(x,y,t)}{\partial y}, \qquad (1)$$
$$u_3(x,y,z,t)=w(x,y,t).$$

In equation (1), the components of displacement inside the plane, along x and y directions are shown by the functions u(x, y) and v(x, y), respectively, and also,w(x, y) is the radial deformation. The parameter z at any point of the plane indicates bending, and $t$shows the time parameter.Changes in elastic strain energy U of an elastic object are expressed as[12]:

$$\delta U=\int_V(\sigma_{ij}\delta\varepsilon_{ij})dV \qquad (2)$$

Parameters σ_{ij} and $\delta\varepsilon_{ij}$ are components of Cauchy stress tensor $\overline{\sigma}$, and $\delta\overline{\varepsilon}$ demonstrates the changes of straintensor. Additionally, the non-zero components of strain based on the Donnel classic theory for the cylindrical shells are as follows [12].

$$\varepsilon_{11}=\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}$$
$$\varepsilon_{22}=\frac{\partial y}{\partial y}+\frac{w}{R}-z\frac{\partial^2 w}{\partial y^2} \qquad (3)$$
$$2\varepsilon_{12}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2z\frac{\partial^2 w}{\partial x\partial y}$$

Under practical conditions, the amount of traction normal force on both the upper and lower sheets is relatively negligible or zero. As a result and due to considering the relatively small thickness of the sheet, the stresscomponent σ_{33} in all parts of the plane, compared to other stress factors, is not significant; so, in order to derivethe formulation of structural stress-strain, the state of plane-stress is applied. For the layered composite, the structural stress-strain relationships for each layer are defined as [12]:

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix}=\overline{Q}_k(\theta)\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{Bmatrix}$$
$$\overline{Q}_k(\theta)=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix} \qquad (4)$$

In the above equations, $\overline{Q}_k(\theta)$ is the stiffness matrix along the non-principal direction, which is expressed as the following [12]:

$$\overline{Q}_k(\theta)=[H(\theta_k)]^T QH_k(\theta_k) \tag{5}$$

Where, Q is the stiffness matrix in the original coordinates, and $H(\theta_k)$ is the transmission matrix for each layerand they are defined as follows [12]:

$$Q=\begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_1}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$
$$H(\theta_k)=\begin{bmatrix} Cos^2(\theta_k) & Sin^2(\theta_k) & Sin(2\theta_k) \\ \frac{\nu_{12}E_1}{1-\nu_{12}\nu_{21}} & Cos^2(\theta_k) & -Sin(2\theta_k) \\ -\frac{Sin(2\theta_k)}{2} & \frac{Sin(2\theta_k)}{2} & Cos(2\theta_k) \end{bmatrix} \tag{6}$$

Governing equations of dynamic equilibrium

In order to extract the governing formulation of the static equilibrium, first,the variation of strain energy and external work are calculated by the central Traction, and then,by replacing them in the principle of minimization of the total potential,the governing equations are achieved. For this reason, the changes in the strain energy of the cylindrical shell are equal to:

$$\delta U_\varepsilon=\int_A Pdxdy \tag{7}$$

Wherein the above:

$$\begin{aligned} P=&-M_{11}\frac{\partial^2(\delta w)}{\partial x^2}-M_{22}\frac{\partial^2(\delta w)}{\partial y^2}\\ &-2M_{12}\frac{\partial^2(\delta w)}{\partial x\partial y}+\frac{N_{22}}{R}\delta w+N_{11}\frac{\partial(\delta u)}{\partial x}\\ &+N_{22}\frac{\partial(\delta v)}{\partial y}+N_{12}\left(\frac{\partial(\delta u)}{\partial y}+\frac{\partial(\delta v)}{\partial x}\right) \end{aligned} \tag{8a}$$

In Eq. (8a), the force and moment stress resultants can be defined as follow:

$$\begin{aligned} N_{11}&=\int_{-h/2}^{h/2}\sigma_{11}dz=A_1(x)\frac{\partial u}{\partial x}+(A_1(x)-2A_2(x))(\frac{\partial v}{\partial y}+\frac{w}{R})\\ N_{22}&=\int_{-h/2}^{h/2}\sigma_{22}dz=(A_1(x)-2A_2(x))\frac{\partial u}{\partial x}+A_1(x)(\frac{\partial v}{\partial y}+\frac{w}{R})\\ N_{12}&=\int_{-h/2}^{h/2}\sigma_{12}dz=A_2(x)(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})\\ M_{11}&=\int_{-h/2}^{h/2}z\,\sigma_{11}dz=-D_1(x)\frac{\partial^2 w}{\partial x^2}-(D_1(x)-2D_2(x))\frac{\partial^2 w}{\partial y^2}\\ M_{22}&=\int_{-h/2}^{h/2}z\,\sigma_{22}dz=-(D_1(x)-2D_2(x))\frac{\partial^2 w}{\partial x^2}-D_1(x)\frac{\partial^2 w}{\partial y^2},\\ M_{12}&=\int_{-h/2}^{h/2}z\,\sigma_{12}dz=-2D_2(x)\frac{\partial^2 w}{\partial x\partial y}. \end{aligned} \tag{8b}$$

Where:

$$\begin{aligned} (A_1(x),D_1(x))&=\frac{E}{1-\nu^2}\left(h(x),\ \frac{(h(x))^3}{12}\right),\\ (A_2(x),D_2(x))&=\frac{E}{2(1+\nu)}\left(h(x),\ \frac{(h(x))^3}{12}\right). \end{aligned} \tag{8c}$$

By applying two integrations by parts to this equation, we will have:

$$\delta U_\varepsilon=\int_A \delta U^{int}dA+\left[\int_{y=0}^{y=2\pi R}\delta U^{bound}dy\right]_{x=0}^{x=L} \tag{9}$$

Thevariations of kinematic energy of the composite cylindrical shell ($\delta(K.E.)$), with a rotating speed, are expressed as:

$$\begin{aligned} \delta(K.E.)=&-\int_A m\begin{pmatrix}\ddot{u}\delta u+(\ddot{v}+2\Omega\dot{w}-\Omega^2 v)\delta v+\\ (\ddot{w}-2\Omega\dot{v}-\Omega^2 w)\delta w\end{pmatrix}dA\\ &+\int_A m\frac{\partial}{\partial t}\begin{pmatrix}\dot{u}\delta u+(\dot{v}+w\Omega)\delta v+\\ (\dot{w}-v\Omega)\delta w\end{pmatrix}dA\\ &+\int_A I\left(\frac{\partial^2\ddot{w}}{\partial x^2}+\frac{\partial^2\dot{w}}{\partial y^2}\right)\delta wdA-\int_0^{R\theta}\left(I\frac{\partial\ddot{w}}{\partial x}\delta w\right)dy\\ &-\int_0^{L}\left(I\frac{\partial\ddot{w}}{\partial y}\delta w\right)dx-\int_A I\frac{\partial}{\partial T}\left(\frac{\partial^2\dot{w}}{\partial x^2}+\frac{\partial^2\dot{w}}{\partial y^2}\right)\delta wdA\\ &+\int_0^{R\theta}\frac{\partial}{\partial t}\left(I\frac{\partial\dot{w}}{\partial x}\delta w\right)dy+\int_0^{L}\frac{\partial}{\partial t}\left(I\frac{\partial\dot{w}}{\partial y}\delta w\right)dx \end{aligned} \tag{10}$$

On the other hand, according to the rotation of the shell,the initial circumferential tensionwill be created, and as a result, the initial strain energy that is stored is calculated as:

$$U_0=\frac{1}{2}\int_A N_0^\theta\left(\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)^2+\left(-\frac{\partial w}{\partial y}+\frac{v}{R}\right)^2\right)dA \tag{11}$$

In the above equation, $N_0^\theta=mR^2\Omega^2$ is the initial circumferential tension, and also, the work done is obtained from:

$$\begin{aligned} U_0=&-\int_A N_0^\theta\begin{pmatrix}\left(\frac{\partial^2 u}{\partial y^2}\right)du+\left(\frac{\partial^2 v}{\partial y^2}+\frac{2}{R}\frac{\partial w}{\partial y}-\frac{v}{R^2}\right)\delta v\\ +\left(\frac{\partial^2 w}{\partial y^2}-\frac{2}{R}\frac{\partial v}{\partial y}-\frac{w}{R^2}\right)\delta v\end{pmatrix}dA\\ &+\int_{x=0}^{x=L}N_0^\theta\left(\left(\frac{\partial u}{\partial y}\right)\delta u+\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)\delta v+\left(\frac{\partial w}{\partial y}-\frac{v}{R}\right)\right)dx \end{aligned} \tag{12}$$

On the other hand, the reaction between the cylindrical shell and the elastic foundation,which is of Winkler type, aremodeled as the external force applied to the shell as:

$$\delta W=-\int_A k_W w(x,y)dA \tag{13}$$

Hamilton's principle expressed as follows in the time period between t_1 and t_2 is noted as:

$$\int_{t_1}^{t_2} (\delta(K.E.) - \delta U + \delta W)\, dt = 0 \qquad (14)$$

By replacing the changes of strain energy, kinetic energy, and the work done, the governing dynamic equilibrium equations are derived using classic Donnel model, as follows:

$$\begin{aligned}
&\delta u:\\
&\frac{\partial N_{11}}{\partial x} + \frac{\partial N_{12}}{\partial y} + N_0^{\theta}\frac{\partial^2 u}{\partial y^2} = m\ddot{u}\\
&\delta v:\\
&\frac{\partial N_{12}}{\partial x} + \frac{\partial N_{21}}{\partial y}\\
&+N_0^{\theta}\left(\frac{\partial^2 v}{\partial y^2} + \frac{2}{R}\frac{\partial w}{\partial y} - \frac{v}{R^2}\right) =\\
&m\left(\ddot{v} + 2\Omega\dot{w} - \Omega^2 v\right)\\
&\delta w:\\
&\frac{\partial^2 M_{11}}{\partial x^2} + \frac{\partial^2 M_{22}}{\partial y^2} + 2\frac{\partial^2 M_{12}}{\partial x \partial y}\\
&-\frac{N_{22}}{R} + N_0^{\theta}\left(\frac{\partial^2 w}{\partial y^2} - \frac{2}{R}\frac{\partial v}{\partial y} - \frac{w}{R^2}\right)\\
&-k_W w = m\left(\ddot{w} - 2\Omega\dot{v} - \Omega^2 w\right)\\
&-I\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right)
\end{aligned} \qquad (15)$$

In addition to this, the boundary conditions at both ends of the shell ($x = 0, L$)are as follows:

$$\begin{aligned}
&N_{11} = 0, \quad \text{or } \delta u = 0\\
&N_{22} + N_0^{\theta}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right) = 0, \quad \text{or } \delta v = 0\\
&\frac{\partial M_{22}}{\partial y} + 2\frac{\partial M_{12}}{\partial x} +\\
&N_0^{\theta}\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right) + I\frac{\partial \ddot{w}}{\partial y} = 0 \text{ or } \delta w = 0\\
&M_{11} = 0, \text{or } \delta\left(\frac{\partial w}{\partial x}\right) = 0
\end{aligned} \qquad (16)$$

Similarly, for the lateral boundaries of the shell,at $y = 0, R\beta$, one has:

$$\begin{aligned}
&N_{12} + N_0^{\theta}\frac{\partial u}{\partial y} = 0 \quad \text{or } \delta u = 0\\
&N_{22} = 0, \quad \text{or } \delta v = 0\\
&\frac{\partial M_{11}}{\partial x} + 2\frac{\partial M_{12}}{\partial y} + I\frac{\partial \ddot{w}}{\partial x} = 0 \quad \text{or } \delta w = 0\\
&M_{11} = 0, \quad \text{or } \delta\left(\frac{\partial w}{\partial x}\right) = 0
\end{aligned} \qquad (17)$$

Finally, the following conditions must be met for the four corners of the shell:

$$M_{12} = 0 \text{ or } \delta w = 0 \qquad (18)$$

Resultants of stress force and moment that are seen in the governing equations and boundary conditions should be calculated.

The governing equations in terms of kinematic parameters

By replacing the stress resultants into the equation (14), the following equation is obtained:

$$\begin{aligned}
&\delta u:\\
&A_{11}\frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 v}{\partial x \partial y} +\\
&A_{12}\frac{1}{R}\frac{\partial w}{\partial x} + A_{66}\frac{\partial^2 u}{\partial y^2} +\\
&N_0^{\theta}\frac{\partial^2 u}{\partial y^2} - m(x)\ddot{u} = 0\\
&\delta v:\\
&A_{22}\frac{\partial^2 v}{\partial y^2} + (A_{12} + A_{66})\frac{\partial^2 u}{\partial x \partial y} +\\
&A_{22}\frac{1}{R}\frac{\partial w}{\partial y} + A_{66}\frac{\partial^2 v}{\partial x^2} +\\
&N_0^{\theta}\left(\frac{\partial^2 v}{\partial y^2} + \frac{2}{R}\frac{\partial w}{\partial y} - \frac{v}{R^2}\right)\\
&-m(x)\left(\ddot{v} + 2\Omega\dot{w} + -\Omega^2 v\right) = 0\\
&\frac{\partial N_{12}}{\partial x} + \frac{\partial N_{21}}{\partial y}\\
&+N_0^{\theta}\left(\frac{\partial^2 v}{\partial y^2} + \frac{2}{R}\frac{\partial w}{\partial y} - \frac{v}{R^2}\right) =\\
&m\left(\ddot{v} + 2\Omega\dot{w} - \Omega^2 v\right)\\
&\delta w:\\
&D_{22}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} +\\
&+D_{22}\frac{\partial^4 w}{\partial y^4} - \frac{1}{R}\left(A_{12}\frac{\partial u}{\partial x} + A_{22}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right)\right)\\
&+N_0^{\theta}\left(\frac{\partial^2 w}{\partial y^2} + \frac{2}{R}\frac{\partial v}{\partial y} - \frac{w}{R^2}\right) - k_w w =\\
&\dot{m}\left(\ddot{w} + 2\Omega\dot{v} - \Omega^2 w\right) - I\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right)
\end{aligned} \qquad (19)$$

Similarly,for the boundary conditions at both ends of the shell ($x = 0, L$), one has:

$$
\begin{aligned}
&A_{11}\frac{\partial u}{\partial x}+A_{12}\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)=0 \text{ or } \delta u=0\\
&A_{66}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=0 \qquad \text{or } \delta v=0\\
&D_{11}\frac{\partial^3 w}{\partial x^3}+(D_{12}+4D_{66})A_{66}\frac{\partial^3 w}{\partial x\partial y^2}\\
&-I\frac{\partial \ddot{w}}{\partial x}=0 \qquad \text{or } \delta w=0\\
&D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}=0 \quad \text{or} \quad \delta\left(\frac{\partial w}{\partial x}\right)=0
\end{aligned}
\tag{20}
$$

Also, for the lateral boundaries of the shell at $y = 0, R\beta$, one has:

$$
\begin{aligned}
&A_{66}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+N_0^{\theta}\frac{\partial u}{\partial y}=0 \qquad \text{or} \quad \delta u=0\\
&A_{12}\frac{\partial u}{\partial x}+A_{22}\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)+\\
&N_0^{\theta}\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)=0 \text{ or } \quad \delta v=0\\
&D_{22}\frac{\partial^3 w}{\partial y^3}+(D_{12}+4D_{66})\frac{\partial^3 w}{\partial y\partial x^2}\\
&-I\frac{\partial \ddot{w}}{\partial y}=0 \qquad \text{or} \quad \delta w=0\\
&D_{12}\frac{\partial^2 w}{\partial x^2}+D_{22}\frac{\partial^2 w}{\partial y^2}=0 \quad \text{or} \quad \delta\left(\frac{\partial w}{\partial y}\right)=0
\end{aligned}
\tag{21}
$$

Solution of equations

Regarding the linear equations based on the expansion of the dual-Fourier, the vibrations of the composite cylindrical shells on elastic foundation are solved. to achievethis aim, displacement components for the annular cylindrical shells under simply-supported conditions are expressed in the following form [9]:

$$
\begin{aligned}
u(x,y)&=u_{mn}\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n}{R}y+\omega_{mn}t\right)\\
v(x,y)&=v_{mn}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n}{R}y+\omega_{mn}t\right)\\
w(x,y)&=w_{mn}\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n}{R}y+\omega_{mn}t\right)
\end{aligned}
\tag{22}
$$

In the above, $\omega_{m,n}$ is the natural frequency of the cylindrical shell.

By substituting displacement componentsinto the governing equations, a linear set of algebraic equations is obtained:

$$
[K]-\Omega^2[G]-\Omega\omega_{mn}[C]-\omega_{mn}{}^2[M]\begin{Bmatrix}u_{mn}\\ v_{mn}\\ w_{mn}\end{Bmatrix} \tag{23}
$$

Where in the above, one has:

$$
\begin{aligned}
&[K]=\begin{bmatrix}k_{11}&k_{12}&k_{13}\\k_{21}&k_{22}&k_{23}\\k_{31}&k_{32}&k_{33}\end{bmatrix},[G]=\begin{bmatrix}g_{11}&0&0\\0&g_{22}&0\\0&0&g_{33}\end{bmatrix}\\
&[C]=\begin{bmatrix}0&0&0\\0&0&C_{23}\\0&C_{32}&0\end{bmatrix},[M]=\begin{bmatrix}M_{11}&0&0\\0&M_{22}&0\\0&0&M_{33}\end{bmatrix}
\end{aligned}
\tag{24}
$$

In the above, the components of the matrices [K], [G], [C] and [M] are defined as:

$$
\begin{aligned}
k_{11}&=-\left(A_{11}\left(\frac{m\pi}{L}\right)^2+A_{66}\left(\frac{n}{R}\right)^2\right)\\
k_{12}&=-\left(\frac{n}{R}\right)\left(\frac{m\pi}{L}\right)(A_{11}+A_{66})\\
k_{12}&=\left(\frac{A_{12}}{R}\right)\left(\frac{m\pi}{L}\right)\\
k_{21}&=\left(\frac{n}{R}\right)\left(\frac{m\pi}{L}\right)(A_{11}+A_{66})\\
k_{23}&=-A_{22}\left(\frac{n}{R^2}\right)\\
k_{31}&=-\left(\frac{A_{12}}{R}\right)\left(\frac{m\pi}{L}\right)\\
k_{32}&=-A_{22}\left(\frac{n}{R^2}\right)\\
k_{33}&=D_{11}\left(\frac{m\pi}{L}\right)^4+2(D_{12}+2D_{66})\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{R^2}\right)\\
&D_{22}\left(\frac{n}{R}\right)^4-\left(\frac{A_{22}}{R^2}\right)-k_W\\
g_{11}&=\tilde{m}n^2\\
g_{22}&=\tilde{m}n^2\\
g_{23}&=2\tilde{m}n\\
g_{33}&=\tilde{m}n^2\\
C_{23}&=-2\tilde{m}\\
m_{11}&=-\tilde{m}\\
m_{22}&=-\tilde{m}\\
m_{33}&=-\tilde{m}-\left(\left(\frac{n}{R}\right)^2+\left(\frac{m\pi}{L}\right)^2\right)I
\end{aligned}
\tag{25}
$$

By equating the determinant of the matrix of coefficients to zero, the frequency equation is achieved:

$$
\begin{vmatrix}
k_1-\Omega^2 g_{11}-\omega_{mn}^2 m_{11} & k_{12} & k_{13}\\
k_{21} & k_{22}-\Omega^2 g_{22}-\omega_{mn}^2 m_{22} & k_{23}-\Omega^2 g_{23}-\Omega\omega_{mn}c_{23}\\
k_{31} & k_{33}-\Omega^2 g_{23}-\Omega\omega_{mn}c_{23} & k_{33}-\Omega^2 g_{33}-\omega_{mn}^2 m_{33}
\end{vmatrix}=0
\tag{26}
$$

The frequency equation can be rewritten as the following:

$$\begin{aligned}&a_1\omega_{mn}^6 + a_2\Omega^6 + a_3\omega_{mn}^4\Omega^2 + a_4\omega_{mn}^2\Omega^4 + \\ &a_5\omega_{mn}^2 + a_6\Omega^4 + a_7\omega_{mn}^2\Omega^2 + a_8\omega_{mn}^2 + a_9\Omega^2 \\ &a_{10}\Omega\omega_{mn} + a_{11} = 0\end{aligned} \tag{27}$$

Where, in the above:

$$\begin{aligned}
&a_1 = -m_{11}m_{22}m_{33} \\
&a_2 = -g_{11}g_{22}g_{33} \\
&a_3 = -g_{11}m_{22}m_{33} - m_{11}(g_{22}m_{33} + m_{22}g_{33}) \\
&a_4 = -m_{11}g_{22}g_{33} - g_{11}(m_{22}g_{33} + g_{22}m_{33}) \\
&a_5 = k_{11}m_{22}m_{33} + m_{11}(k_{22}m_{33} + m_{22}k_{33}) \\
&a_6 = k_{11}g_{22}g_{33} + g_{11}(k_{22}g_{33} + g_{22}k_{33}) \\
&a_7 = k_{11}(g_{22}m_{33} + m_{22}g_{33}) + \\
&g_{11}(k_{22}m_{33} + m_{22}k_{33}) + m_{11}(k_{22}g_{33} + g_{22}k_{33}) \\
&a_8 = -k_{11}(k_{22}m_{33} + m_{22}k_{33}) - \\
&m_{11}k_{22}k_{33} + m_{33}k_{12}k_{21} + m_{22}k_{13}k_{31} \\
&a_9 = -k_{11}(k_{22}g_{33} + g_{22}k_{33}) - g_{23}k_{12}k_{31} + \\
&g_{33}k_{12}k_{21} - g_{23}k_{21}k_{13} + g_{22}k_{13}k_{31} \\
&a_{10} = -c_{23}(k_{12}k_{31} + k_{21}k_{13}) \\
&a_{11} = k_{11}k_{22}k_{33} + k_{12}(k_{31}k_{23} - k_{21}k_{33}) + \\
&k_{13}(k_{21}k_{32} - k_{31}k_{23})
\end{aligned} \tag{28}$$

Finite element modeling

The finite element model includes a rotating annular cylindrical shell based on the elastic foundation.
The mechanical properties of the shell are included in Table 1. Fiber angles are considered to be $[0,90,0,90]_s$.

Table 1. Mechanical properties of the composite.

S-glass/epoxy				
$E_1(Gpa)$	$E_2(Gpa)$	$G_{12}(Gpa)$	ν_{12}	$\rho(kg/m^3)$
43.5	11.58	3.45	0.27	1700

Both sides of the cylinder are regarded as simply-supported. Also, the shell has rotational speed around its central axis.The S4R element isused for gridding the composite cylindrical shell.

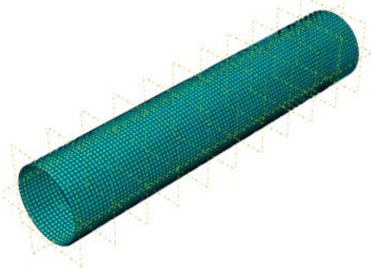


Figure 2. Finite element model of the composite cylindrical.

Verification

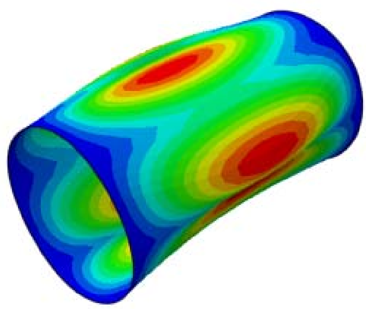
Here in this section, numerical and analytical results for the composite cylindrical shells based on the elastic foundation are provided. The thickness of the shell is considered to be *8 mm*. Also, the values of the natural frequencies and the rotation of the shell have been non-dimentionalized by means of $\omega^* = \frac{1}{R}\sqrt{E/\rho}$ in the diagrams.

In Table 2, the values of the smallest dimensionless natural frequency, ω/ω^*, for the rotational speed of $\Omega = \omega^*$ achieved from the analytical solution, have been presented and compared with the results obtained by the finite element software, ABAQUS, fordifferent geometric ratios.

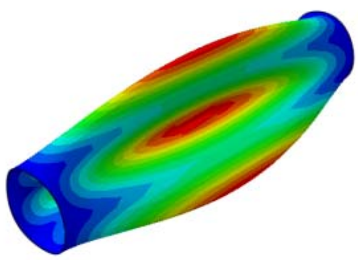
Table 2. Comparison of dimensionless values of the smallest dimensionless natural frequency$\omega/\omega^*$for $L/R = 10$.

Ω/ω^*			
	Present	***FEM***	***Difference (%)***
0.5	1.438	1.33	7.5
1	2.952	2.71	8.1
1.5	4.783	4.223	11.7

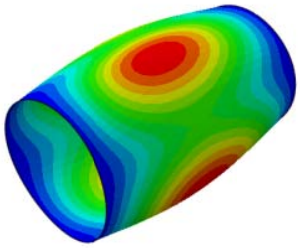
The results indicate that the analytical method is of acceptableaccuracy. In Fig. 3, the contours achieved from the finite element method arepresented.



L/R=10and $\Omega/\omega^* = 0.5$



L/R=10and $\Omega/\omega^* = 1$



L/R=10and $\Omega/\omega^* = 1.5$

Figure 3. Results of the numerical solution, with R / h = 125 and various Ω/ω^*.

Analysis of results

In this section, the results of the half-analytical solutions for the free vibration of the annular cylindrical shells on elastic foundation, which were presented in the previous chapter, areprovided. The results are obtained for the values of *m=1* and *n=4*.

The results are plotted as curves of thevalues of dimensionless natural frequencies of the shell versus values of the rotationalex citation frequencies. It is worth mentioning that the diagrams are plotted for positive excitation angular velocity (*Forward whirling*) and also negative (*Back ward whirling*). The composite material of the cylindrical shell is considered to be as of the mechanical properties listed in Table (1).

Table 3. Geometric specifications of the cylindrical composite with lay-up $[0,90,0,90]_s$.

$h_t\,(mm)$	$L(mm)$	$R(mm)$
8	10000	1000

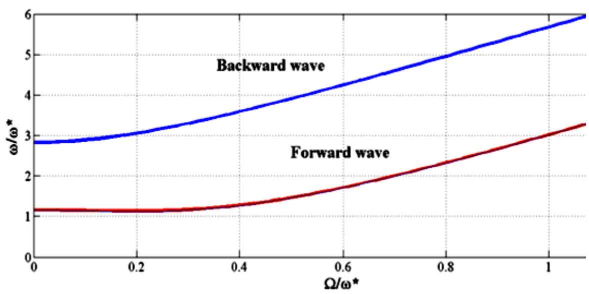
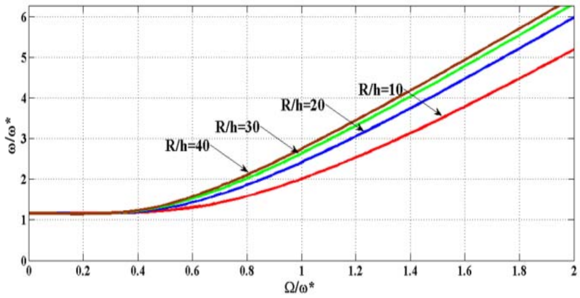


Figure 4. Diagrams of the dimensionless natural frequencies withthe dimensionless excitation frequency of rotation for the shell, with following geometries: (a) $L / R = 10$ and $R / h = 10$ and $K_w = 0.1$ * K_{star} and m = *1* andn = *4.*

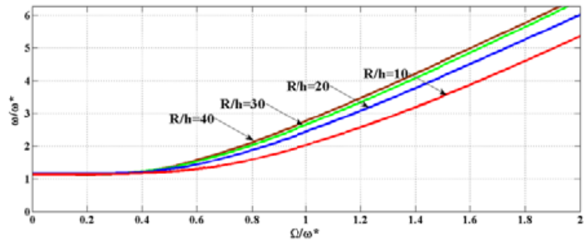
As is clear from the results, thebackward mode values are greater than forward mode. Also, the ratio of ω/ω^* increases non-linearly with the increase of Ω/ω^*. Later in the paper, the effect study of the changes in various parameters on the natural frequency is conducted.

Effect of radius to thickness ratio

Considering $m = 1$, $n = 4$, $L / R = 10$ and $K_w = 10000$ * K_{star}, the changes of ω/ω^* versus Ω/ω^* areplotted for two cases of *Backward* and *Forward*(as in Fig. 5). As is seen, by increasing the amount of *R/h*for the same Ω/ω^*, the amount of ω/ω^* increases. Also, by increasing Ω/ω^*, first the diagram has a linear state and then,it will have an increasing trend.



(a)

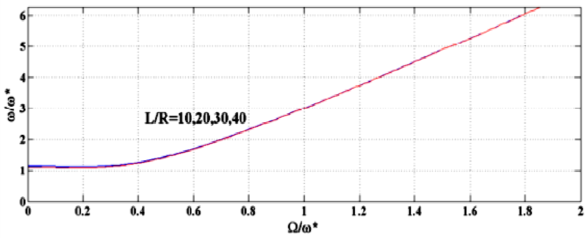


(b)

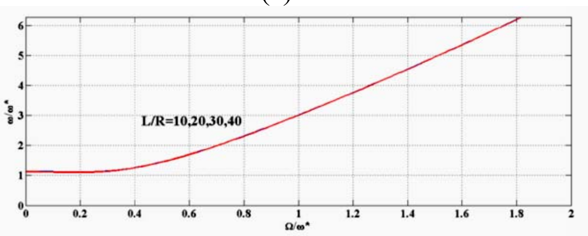
Figure 6. The effect of R/h on the dimensionless natural frequencies versus the dimensionless rotational excitation frequencyfor the composite cylindrical shell (a) *Forward*and (b) *Backward*whirling.

Effect of length to radius ratio:

Considering $m = 1$, $n = 4$, $R/h=10$ and $K_w = 10000$ * K_{star}, changes of ω/ω^* versus Ω/ω^* have beendrawn for the two *Backward* and *Forward* states (as in Fig.7). As is clear, with the increase inthe amount of *L/R*at the constantamount of Ω/ω^*, the value of ω/ω^* increases. Also, by increasing the Ω/ω^*, first the diagram has a linear stateand then, it will have an increasingtrend. As is seen, in the two modes of *Backward* and *Forward,* at different *R/L*, and fixed Ω/ω^*, there is no obvious change noticed in the values of the diagram.



(a)

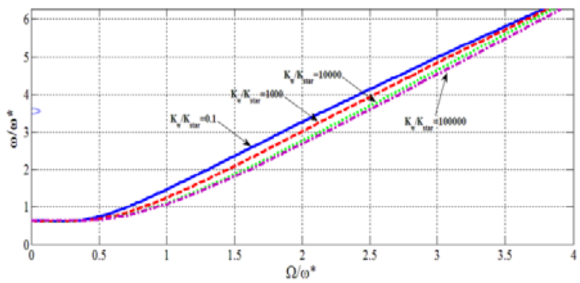


(b)

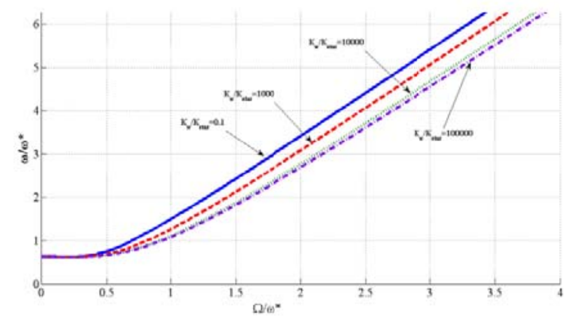
Figure 7. The effect of L/R on the dimensionless natural frequencies versus the dimensionless rotational excitation frequency for the composite cylindrical shell(a)*Forward*and(b) *Backward*whirling.

Effect of stiffnessof the elastic foundation

Considering $m = 1, n = 4$ and $L / R = 10,$ changes of ω/ω^* versus Ω/ω^* areplotted for the two *Backward* and *Forward*states. As shown, by increasing the amount of$K_w = 10000 * K_{star}$at the same amount of Ω/ω^*, the value of ω/ω^* is reduced. Also, by increasing the Ω/ω^*, first the diagram is in the linear state, and later, it will increase.



(a)

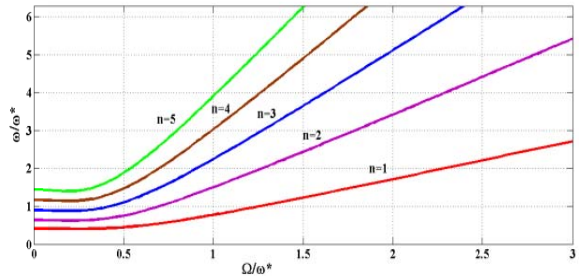


(b)

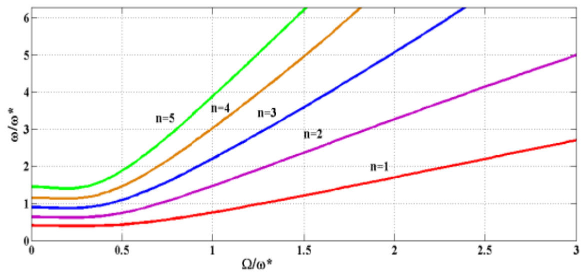
Figure 8. The effect of the elastic foundation on the dimensionless natural frequencies versus the dimensionless rotational excitation frequency for the composite cylindrical shell (a)*Forward*and(b) *Backward*whirling.

Effect of parameter *n*

Considering $m = 1, K_w = 0.1 * K_{star}, R / h = 10$ and $L / R / 10,$changes of ω/ω^* versus changes of parameter n areplotted for the two *Backward* and *Forward*states. As shown, by increasing the value of $n$at the same amount of Ω/ω^*, the value of $\omega/\omega^*$is increased. Also, by increasing Ω/ω^*, the values of ω/ω^*at a similar Ω/ω^*will have more discrepancies.



(a)

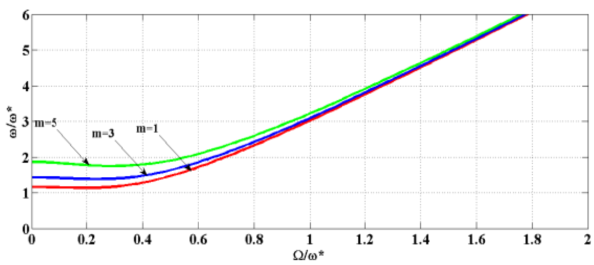


(b)

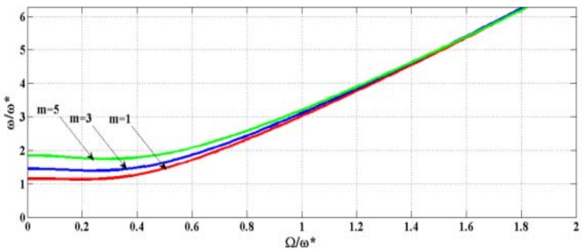
Figure 9. The effect of n on the dimensionless natural frequencies versus the dimensionless rotational excitation frequency for the composite cylindrical shell (a) *Forward*and (b) *Backward*whirling.

Effect of parameter *m*

Considering $n = 4$, $K_w = 0.1 * K_{star}$,$R / h = 10$ and $L / R = 10$,changes of ω/ω^* versus changes of Ω/ω^* areplotted for the two *Backward* and *Forward*states. As shown, by increasing the value of m at the same amount of Ω/ω^*, the value of ω/ω^* is increased. Also, by increasing theΩ/ω^*, the values of ω/ω^* at a similar Ω/ω^*will have fewer discrepancies with each other. Also, the values of the *Backward*state are more than the values of the *forward* state.



(a)

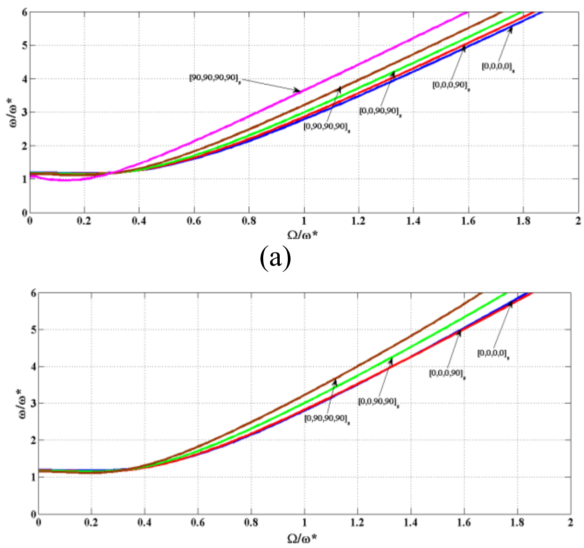


(b)

Figure 10. The effect of m on the dimensionless natural frequencies versus the dimensionless rotational excitation frequency for the composite cylindrical shell (a) *Forward*and (b) *Backward*whirling.

Effect of fiber angle

Regarding$m = 4$, $n = 4$, $K_w = 0.1 * K_{star}$,$R / h = 10$ and $L / R = 10$,changes ofω/ω^* versus changes of Ω/ω^* areplotted for the two *Backward* and *Forward*states. As shown, by increasing the value of fiber angles towards 90 degrees, at the two states of *Backward* and *Forward*, the values of the frequency will increase.



(a)

(b)

Figure 11. The effect of the fiber angle on the dimensionless natural frequencies versus the dimensionless rotational excitation frequency for the composite cylindrical shell (a) *Forward*and (b) *Backward*whirling.

Final conclusion

In this article, the impact of the main parameters on the vibrations of the annular composite cylindrical shells based on the elastic foundation, and rotating with a fixed angular velocity wereinvestigatedusingthe analytical method. The results were extracted and achieved for the two states of positive excitation angular velocity (*forward whirling*) and negative excitation angular velocity (*Backwardwhirling*).

In our vibration analysis of the annular cylindrical shell,the following results were noticed:

- *Backward* mode values are higher than those of the*forward* mode.
- The ω/ω^* value increases non-linearly as the value of Ω/ω^* increases.
- By increase in the amount of *R/h* and *L/R* at the constant value of Ω/ω^*, the value of ω/ω^* is increased.
- By increasing the amount of$K_w = 10000 * K_{star}$ at the constant Ω/ω^*, the value of ω/ω^* is reduced.
- By increasing the values of *n* and *mat*, the constant value of Ω/ω^*, the value of ω/ω^* is increased.
- By increasing fiber angle towards 90 degrees in both *Backward* and *Forward*states, the values of the frequencyareincreased.
- The finite element analysis results are in good matching with the analytical results.

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