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In this paper, with the aim of estimating internal dynamics matrix of a gimbaled Inertial Navigation System (as a discrete linear system), the discrete-time Hamilton-Jacobi-Bellman (HJB) equation for optimal control has been extracted. Heuristic Dynamic Programming algorithm (HDP) for solving the equation has been presented and then a neural network approximation for cost function and control input has been extracted to simplify the solution of HJB. Design process of the optimal controller shows that, we do not need to know the system matrix. This important issue and the convergence of the HDP algorithm to the optimal control policy makes possible the estimation of the internal dynamics matrix.

Keywords: optimal control, Heuristic Dynamic Programming, neural network approximation, gimbaled Inertial Navigation System, Initial Alignment

INTRODUCTION

Initial Alignment in gimbaled Inertial Navigation Systems (INS) is a complicated experimental process, which needs a multivariable control system [1, 2]. Because of uncertainties in the mechanical and electrical parts of INS, sometimes we need to use certain identification methods to determine system parameters [3-6]. As shown in Figure 1, a gimbaled INS consists of a three-axial suspension mechanism, with an assembly of inertial sensors (3 gyroscopes and 3 accelerometers) on it, and some necessary actuators. The design of a control system for this purpose, needs the internal dynamics matrix of the platform. In the present article, we use a new identification method to determine this internal dynamics matrix based on a HDP algorithm.

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Figure 1. a gimbaled INS [1]

System identification is the process of developing or improving the mathematical representation of a physical system using experimental data. There are three types of identification techniques: modal parameter identification, structural model identification and control model identification [7]. The main aim of system identification is to determine a mathematical
Consider an affine dynamic system of the form:
\[ x_{k+1} = f(x_k) + g(x_k)u(x_k) \]  
where \( x \in \mathbb{R}^n, f(x) \in \mathbb{R}^n, g(x) \in \mathbb{R}^{n \times m} \) and the input \( u \in \mathbb{R}^m \).

It is desired to find the control input which minimizes the cost function given as (2):
\[ V(x_k) = \sum_{n=k}^{\infty} x_n^T Q x_n + u_n^T R u_n \]

where \( Q > 0 \) and \( R > 0 \).

From Bellman’s optimality theory and [13]:
\[ u^*(x_k) = \frac{1}{2} R^{-1} g(x_k)^T \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \]  
and:
\[ V^*(x_k) = x_k^T Q x_k + \frac{1}{2} R^{-1} g(x_k)^T \frac{\partial V(x_{k+1})}{\partial x_{k+1}} + V^*(x_{k+1}) \]

This equation reduces to the Ricatti equation in the linear quadratic regulator (LQR) instance, which can be solved with precision. In the general nonlinear case, the HJB cannot be solved exactly.

From [9], the value iteration HDP algorithm for solving it is proposed as:
\[ u_i(x_k) = \text{arg min } u \{ x_k^T Q x_k + u^T R u + V_i(f(x_k) + g(x_k)u) \} \]
\[ = \frac{1}{2} R^{-1} g(x_k)^T \frac{\partial V(x_{k+1})}{\partial x_{k+1}} + V_i(x_{k+1}) \]

where:
\[ V_i(x_{k+1}) = \min_u [x_k^T Q x_k + u^T R u + V_i(x_{k+1})] = x_k^T Q x_k + u_i(x_k) + V_i(f(x_k) + g(x_k)u_i(x_k)) \]

Subsequently, for solving the equations (5), (6), neural network estimation has been proposed as follows:
\[ \hat{V}_i(x) = \sum_{j=1}^L w_{vi} \phi_j(x) = W_{vi} \phi(x) \]  
\[ \hat{u}_i(x) = \sum_{j=1}^M w_{ui} \sigma_j(x) = W_{ui} \sigma(x) \]

where \( \phi(x) \) and \( \sigma(x) \) are the activation functions and \( W_{vi} \) and \( W_{ui} \) are the network weights.

The updated law for \( W_{vi} \) and \( W_{ui} \) is obtained from (9), (10) as:
\[ W_{vi}(x_{k+1}) = \left( \int_{\Omega} \phi(x_k) \phi(x_k)^T d\Omega \right)^{-1} \int_{\Omega} \phi(x_k) \hat{V}_i(x_k, W_{vi}) dx \]
\[ W_{ui}(x_{k+1}) = W_{ui} - \alpha \sigma(x_k)(2R \hat{u}_i(x_k, W_{ui}) + g(x_k)^T \frac{\partial V(x_{k+1})}{\partial x_{k+1}} W_{vi})^T \]

where \( \Omega \) is the training set, \( \alpha \) is a positive step size, and \( m \) is the iteration number of the LMS algorithm.

Assuming the convergence of HDP algorithm for a linear system, the last relations show, \( V_i(x_k) \) and \( u_i(x_k) \) converge to the cost function of the optimal control problem and to the corresponding optimal control input, respectively.

The most important feature which is seen in (9) and (10) is that \( f(x_k) \) isn’t not needed to update the critic neural network weights and this issue makes possible the estimation of system inner dynamics matrix using composition of heuristic dynamic programming control algorithm and neural network.
**ESTIMATION of INNER DYNAMICS MATRIX of DISCRETE-TIME LINEAR SYSTEM**

Consider the linear discrete system (for example with five states), described as:

\[ x(k + 1) = Gx(k) + Bu(k) \]  
\[(11)\]

From LQR method:

\[ u^*(x) = -Lx \]  
\[(12)\]

where:

\[ L = (R + B^TPB)^{-1}B^TPG \]  
\[(13)\]

And \( P \) is the symmetric positive definite of below Ricatti equation:

\[ P = G^TPG + Q - G^TPB(R + B^TPB)^{-1} \times B^TPG ; \]  
\[(14)\]

and

\[ V^*(x) = x^TPx \]  
\[(15)\]

A reasonable selection of activation functions vector for cost and control input would be as:

\[ \Phi(x) = [x_1^2 \ x_2 \ x_3 \ x_1x_2 \ x_1x_3 \ x_2x_3 \ x_2^2 \ x_3^2] \]  
\[(16)\]

Then, converged weight vectors have the form of:

\[ W_p = [w_{p,1}w_{p,2}w_{p,3}w_{p,4}w_{p,5}w_{p,6}w_{p,7}] \]  
\[w_{p,8}w_{p,9}w_{p,10}w_{p,11}w_{p,12}w_{p,13}w_{p,14}w_{p,15}] \]

\[ W_u = \begin{bmatrix} w_{u,1} & w_{u,2} & w_{u,3} & w_{u,4} & w_{u,5} \\ w_{u,2} & w_{u,3} & w_{u,4} & w_{u,5} & w_{u,6} \\ w_{u,3} & w_{u,4} & w_{u,5} & w_{u,6} & w_{u,7} \\ w_{u,4} & w_{u,5} & w_{u,6} & w_{u,7} & w_{u,8} \\ w_{u,5} & w_{u,6} & w_{u,7} & w_{u,8} & w_{u,9} \end{bmatrix} \]
\[(17)\]

and:

\[ \Phi(x) = W^p_\Phi \Phi(x) \]  
\[ \Phi^T(x) = W^u_\Phi \Phi(x) \]

As explained in section 3, HDP algorithm converges to the cost function of the optimal control problem and to the corresponding optimal control input, respectively, i.e.

\[ V^*(x) = V^* \]  
\[ u^*(x) = u^* \]
\[(19)\]

Therefore:

\[ W^p_\Phi \Phi(x) = x^TPx \]  
\[ W^u_\Phi \Phi(x) = -Lx \]
\[(20)\]

We define:

\[ W = \begin{bmatrix} w_{v,1} & 0.5w_{v,2} & 0.5w_{v,3} & 0.5w_{v,4} & 0.5w_{v,5} \\ 0.5w_{v,6} & w_{v,7} & 0.5w_{v,8} & 0.5w_{v,9} & 0.5w_{v,10} \\ 0.5w_{v,11} & 0.5w_{v,12} & w_{v,13} & 0.5w_{v,14} \end{bmatrix} \]
\[(21)\]

Then, (16-21) results:

\[ P = W \]
\[ L = -W_u^T \]
\[(22)\]

Using (13), (22):

\[ -W_u^T = (R + B^T W)B^T W \]
\[(23)\]

If the number of states is equal to control inputs, i.e. size(G)=size(B) and B is nonsingular, (constrains of proposed estimation algorithm), then, there is a unique solution for G from (23):

\[ G = (W^{-1} B^{-1} R + B) W_u^T \]
\[(24)\]

That can be simplified to:

\[ G = (W^{-1} B^{-1} R + B) W_u^T \]
\[(25)\]

A block-diagram of the proposed estimation method is shown in Fig. 2.

**ESTIMATION of INNER DYNAMICS MATRIX of GIMBALED INERTIAL NAVIGATION SYSTEM**

Gyrosopes random error is defined as a stable stochastic process with autocorrelation function as follows [1]:

\[ G = (W^{-1} B^{-1} R + B) W_u^T \]
\[(26)\]

where \( \sigma_i \) is the standard deviation of random error of gyroscopes, which is known, and \( \mu_i \) is the damping coefficient.

Differential equation of gyroscope error is as:

\[ 1 + \mu_iD_i = \sqrt{2\sigma_i^2\mu_i}\; w \]

where \( w \) is a white noise with \( N(0,1) \).
Dynamic equations of inertial navigation system considering gyroscopes disturbances are [14]:

\[
\dot{x}(t) = Gx(t) + Dw(t)
\]

where:

\[
G = \begin{pmatrix}
-\Omega_d & 0 & 0 & 0 & 0 \\
-\Omega_d & 0 & 0 & 0 & 0 \\
0 & 0 & -\mu_E & 0 & 0 \\
0 & 0 & 0 & -\mu_N & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2\sigma_E^2 \mu_E} & 0 & 0 \\
0 & 0 & 0 & \sqrt{2\sigma_N^2 \mu_N} & 0
\end{pmatrix}
\]

In practice two-channel control for horizontal axes is used. While in this paper, for extracting matrix A, because of constrains in the algorithm, a five-channel virtual control is utilized. Therefore:

\[
\dot{x}(t) = Gx(t) + Bu(t) + Dw(t)
\]

By assumption of matching condition \( Dw(t) = BW(t) \), the equation (30) can be rewritten as:

\[
\dot{x}(t) = Gx(t) + Bu(t) + \beta Dw(t)
\]

where:

\[
\beta(t) = u(t) + w(t)
\]

For implementing the proposed method, we need the discrete model of the system. For this aim, we use the approximation:

\[
\dot{x}(t) \approx \frac{x(k+1) - x(k)}{\Delta t}
\]

Therefore:

\[
x(k + 1) = \tilde{G}x(k) + \tilde{B}u(k)
\]

where:

\[
\tilde{G} = I + G\Delta t , \quad \tilde{B} = B\Delta t
\]

If we choose the following parameters:

\[
\Omega_d = \Omega_e \sin(\varphi) \left( \frac{\text{rad}}{s} \right)
\]

\[
\Omega_n = \Omega_e \cos(\varphi) \left( \frac{\text{rad}}{s} \right)
\]

\[
\Omega_e = 7.29 \times 10^{-5} \left( \frac{\text{rad}}{s} \right)
\]

\[
\varphi = 35.5^\circ , \quad \mu_E = 3 \times 10^{-3}
\]

\[
\mu_N = 10^{-3} , \quad \Delta t = 0.01 \text{ (s)}
\]

With substituting in (29), (35):

\[
\tilde{G} = \begin{pmatrix}
1 & 4.23e-7 & 0 & 0 & 0 \\
1 & 4.23e-7 & 0 & 0 & 0 \\
0 & 5.93e-7 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(36)

With applying HDP+NN, \( W_u \) and \( W_e \) converge to:

\[
W_u = \begin{pmatrix}
-1.0000 & 0.0000 & 0.0002 & 0 & 0 \\
-0.0002 & -1.0003 & 0.0000 & 0 & 0 \\
0.0002 & -0.0000 & -1.0002 & 0 & 0 \\
0 & 0 & 0 & -0.00001 & 0 \\
0 & 0 & 0 & 0 & -0.00003
\end{pmatrix}
\]

(37)

The convergence of the two weights (for example \( W_u(3,3) \) and \( W_e(1,2) \)) is shown in Fig. 2.

\[
W_e = \begin{pmatrix}
1562640609 & 273 & -646052 & 0 & 0 \\
1563094323 & 460 & 0 & 0 & 1562864141 \\
0 & 0 & 16666 & 0 & 5
\end{pmatrix}
\]

(39)

Comparing (36) and (39) shows that the proposed estimation method works properly. In [13], another closed-loop subspace identification method which is based on the least-square problem is presented. In this procedure, \( \tilde{G} \) can be found as the solution to the problem

\[
U_n(1: (f - 1)n_y): \tilde{G} = U_n(n_y + 1: f n_y)
\]

(40)

For details see [13].
Extracting Dynamics Matrix of Alignment Process for a Gimbaled …

From (40):
\[
\tilde{G} = \begin{pmatrix}
1.0000 & 1.22e-7 & -0.0002 & 0 & 0 \\
-1.22e-7 & 1.0000 & 0 & 0 & 0 \\
0.0000 & -0.0000 & 0.9999 & 0 & 0 \\
0 & 0 & 0 & 0.98 & 0.98 \\
0 & 0 & 0 & 0 & 0.98
\end{pmatrix}
\] (41)

comparing (39) and(41) shows that the proposed method in this paper enjoys a higher accuracy especially in estimating the drift filter of gyroscopes that is related to the last two entries on the diagonal \(\tilde{G}\).

CONCLUSION

In this paper, a new method for extracting the inner dynamics matrix for a discrete time system was proposed. This idea is obtained from solving the HJB through the composition of Heuristic Dynamic Programming and Neural Network, in which the linear system critic network converges to the solution of ARE, the control network converges to the optimal policy and the internal dynamics did not require to implement HDP. This method was applied for estimating the internal dynamics matrix of a gimbaled Inertial Navigation System. Furthermore, the estimation of gyroscopes random error filter was performed with precision.

REFERENCES

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