JAST. Vol. 9, No. 2, pp 25-32

JAST Journal of Aerospace Science and Technology

Approximate Solution of Sensitivity Matrix of Required Velocity Using Piecewise Linear Gravity Assumption

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In this paper, an approximate solution of sensitivity matrix of required velocity with final velocity constraint is derived using a piecewise linear gravity assumption. The total flight time is also fixed for the problem. Simulation results illustrate the accuracy of the method. Increasing the midway points for linearization, enhances the accuracy of the solution, which in turn, depends on the total flight time.

1 Introduction

Required velocity and its sensitivity matrix are well-known concepts in space applications. The required velocity is a hypothetical instantaneous velocity for a space vehicle to reach a set of orbital/final constraints only under gravitational acceleration. Depending on orbital/final constraints, different required velocities can be defined. The sensitivity matrix is the partial derivative of the required velocity with respect to position vector [1-5].

The analytical solutions of the required velocity and its sensitivity matrix have been obtained for several orbital/final constraints, such as an elliptic orbit with specified semi-major axis and eccentricity, final position vector constraint with specified energy, and a specified flight direction at the target position [3]. Since analytical/explicit solution could not be obtained for certain orbital/final constraints, approximate or iterative algorithms have been developed for the problem.

The well-known Q-guidance scheme presented in Ref. [3] is workable if it is possible to define an instantaneous required velocity meeting mission objectives which is only a function of current position, as stated by Battin [3]. This requirement cannot be met for the problem having both final position vector and velocity constraints. In 1987, Bhat and Shrivastava developed a modified Q-guidance scheme for placing a payload into a specified circular orbit [6].

By defining two required velocities and their corresponding sensitivity matrices, the well-known Q-guidance scheme can be modified for a problem that has final position vector and velocity constraints with specified final time [7]. The first required velocity is defined as the required velocity to attain a desired final position in a specified final time, whereas the final constraint for the second required velocity is the final velocity.

A lot of research can be found in the literature for the first required velocity and its sensitivity matrix, but a few works are available for the second required velocity and its sensitivity matrix [8]. Several methods may be utilized to solve the problem analytically, such as perturbation techniques and time-varying gravity assumption. Also, piecewise linear assumption is an appropriate approach to deal with a group of nonlinear problems in mathematics, guidance, and control theory [9-13].

In the previous work, an approximate solution of sensitivity matrix with final velocity constraint has been presented using linear gravity assumption along the path of current position to the final position vectors [14]. However, the final position is not fixed in this problem and must be obtained for the solution. Higher accuracy levels can be achieved using piecewise linear gravity assumption which will be the subject of the present work.

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2 Basic Formulation

Consider a spacecraft modeled as a particle P moving only under gravitational acceleration. The governing equation of motion is $\dot{\mathbf{v}} = \mathbf{g}(\mathbf{r})$; where $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{g}$ are the spacecraft position, velocity, and gravitational acceleration with respect to an inertial reference, respectively; and the overdot represents the differentiation with respect to time t. Integrating the preceding relation twice yields:

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{g}(\mathbf{r}(\xi))\, d\xi \tag{1a}$$

$$\mathbf{r}(t) = \mathbf{r}_0 + (t - t_0)\mathbf{v}_0 + \int_{t_0}^{t} (t - \xi)\,\mathbf{g}(\mathbf{r}(\xi))\, d\xi \tag{1b}$$

where the subscript "0" denotes the initial value. Therefore, the position and velocity at an arbitrary time $t_k > t_0$ are given by:

$$\mathbf{v}(t_k) = \mathbf{v}_0 + \int_{t_0}^{t_k} \mathbf{g}(\mathbf{r}(t))\, dt \tag{2a}$$

$$\mathbf{r}(t_k) = \mathbf{r}_0 + (t_k - t_0)\mathbf{v}_0 + \int_{t_0}^{t_k} (t_k - t)\,\mathbf{g}(\mathbf{r}(t))\, dt \tag{2b}$$

For the final time t_f, it suffices if t_k is replaced by t_f.

The above equations are not integrable in the current form, so the time interval of $[t_0, t_f]$ is divided into N subintervals of $[t_j, t_{j+1}]$ having time length of $T_{j+1} = t_{j+1} - t_j$ for $j = 0,1,\ldots,N-1$ (see Fig.1). An approximate solution can be found using linear gravity assumption for each interval of $[t_j, t_{j+1}]$, that is,

$$\mathbf{g}(\mathbf{r}(t)) = \frac{t_{j+1} - t}{t_{j+1} - t_j}\mathbf{g}(\mathbf{r}(t_j)) + \frac{t - t_j}{t_{j+1} - t_j}\mathbf{g}(\mathbf{r}(t_{j+1})) \tag{3}$$

However, in a more general form, nonlinear functions between two or more points may be selected. This is a trade-off among complexity, accuracy, and computational burden.

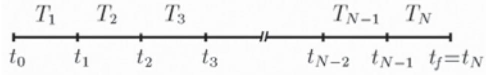


Figure 1. Time intervals representation.

3 Linear Approximation of Velocity

The integral part of Eq. (2a) may be broken down into the summation of integrals:

$$\mathbf{v}(t_f) = \mathbf{v}_0 + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \mathbf{g}(\mathbf{r}(t))\, dt \tag{4}$$

Substitution of Eq. (3) into the preceding relation and integration results in

$$\mathbf{v}(t_f) = \mathbf{v}_0 + \frac{1}{2}T_1\mathbf{g}_0 + \frac{1}{2}T_N\mathbf{g}_N + \frac{1}{2}\sum_{j=1}^{N-1}(T_j + T_{j+1})\mathbf{g}_j \quad \text{for } N \geq 2 \tag{5}$$

where $\mathbf{g}_j$ stands for $\mathbf{g}(\mathbf{r}(t_j))$.

For the special case of fixed-time interval $(T_j = T)$ we have:

$$\mathbf{v}(t_f) = \mathbf{v}_0 + \frac{1}{2}T(\mathbf{g}_0 + 2\mathbf{g}_1 + \cdots + 2\mathbf{g}_{N-1} + \mathbf{g}_N) \tag{6}$$

To obtain $\mathbf{g}_j$'s for computation of the final velocity, the position vectors at the time t_j's must be approximated.

4 Linear Approximation of Position

To disintegrate the equation of position vector, Eq. (2b) is written in the following summation form:

$$\mathbf{r}_k = \mathbf{r}_0 + (t_k - t_0)\mathbf{v}_0 + \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - t)\,\mathbf{g}(\mathbf{r}(t))\, dt \tag{7}$$

where $\mathbf{r}_k$ denotes $\mathbf{r}(t_k)$. Using the piecewise linear gravity (3) we obtain:

$$\int_{t_j}^{t_{j+1}} (t_k - t)\,\mathbf{g}(\mathbf{r}(t))\, dt = \frac{1}{6}T_{j+1}[3(t_k - t_{j+1}) + 2T_{j+1}]\mathbf{g}_j + \frac{1}{6}T_{j+1}[3(t_k - t_j) - 2T_{j+1}]\mathbf{g}_{j+1} \tag{8}$$

We can thus write $(2 \leq k \leq N)$

$$6\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - t)\,\mathbf{g}(\mathbf{r}(t))\, dt = T_1[3(t_k - t_1) + 2T_1]\mathbf{g}_0 + T_k^2\mathbf{g}_k + \sum_{j=1}^{k-1} [T_j^2 - T_{j+1}^2 + 3(T_j + T_{j+1})(t_k - t_j)]\mathbf{g}_j \tag{9}$$

For $k = 1$ we have

$$\mathbf{r}_1 = \mathbf{r}_0 + T_1\mathbf{v}_0 + \frac{1}{6}T_1^2(2\mathbf{g}_0 + \mathbf{g}_1) \tag{10}$$

In the case of fixed-time interval, we have:

$$t_k - t_j = (k - j)T, \quad \text{for} \quad 0 \leq j < k \leq N \tag{11}$$

Therefore, Eq. (9) simplifies to $(2 \leq k \leq N)$:

$$6\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_k - t)\mathbf{g}(\mathbf{r}(t))\,dt = T^2(3k-1)\mathbf{g}_0 + T^2\mathbf{g}_k + 6T^2\sum_{j=1}^{k-1}(k-j)\mathbf{g}_j \quad (12)$$

5 Required Velocity with Velocity Constraint

The required velocity of a spacecraft to reach the final desired velocity of $\mathbf{v}_f^*$ under gravity acceleration in a specified final time t_f is denoted by $\mathbf{v}_v^*$. Without loss of generality, we formulate the problem for $\mathbf{v}_v^*(t_0)$.

Using Eq. (1a) the required velocity $\mathbf{v}_v^*(t_0)$ is given by:

$$\mathbf{v}_v^*(t_0) = \mathbf{v}_f^* - \int_{t_0}^{t_f}\mathbf{g}[\mathbf{r}(\mathrm{t})\,|_{\mathbf{v}_0=\mathbf{v}_v^*(t_0)}]dt \quad (13)$$

Using piecewise linear approximation results in:

$$\mathbf{v}_v^*(t_0) = \mathbf{v}_f^* - \frac{1}{2}(T_1\mathbf{g}_0 + T_N\mathbf{g}_N) - \frac{1}{2}\sum_{j=1}^{N-1}(T_j + T_{j+1})\mathbf{g}_j \quad (14)$$

For the fixed-time interval, we have:

$$\mathbf{v}_v^*(t_0) = \mathbf{v}_f^* - \frac{1}{2}T(\mathbf{g}_0 + 2\mathbf{g}_1 + \cdots + 2\mathbf{g}_{N-1} + \mathbf{g}_N) \quad (15)$$

As can be seen in the two preceding relations, first $\mathbf{g}_j$'s or $\mathbf{r}_j$'s are to be calculated. Several approaches may be utilized for this purpose. The simplest method is that, $\mathbf{r}_k$ is calculated for constant gravity assumption; that is,

$$\mathbf{r}(t) = \mathbf{r}_0 + (t-t_0)\mathbf{v}_v^*(t_0) + \frac{1}{2}\mathbf{g}_0(t-t_0)^2 \quad (16)$$

Substitution of $\mathbf{v}_v^*(t_0) = \mathbf{v}_f^* - \mathbf{g}_0(t_f - t_0)$ into the preceding relation yields:

$$\mathbf{r}(t) = \mathbf{r}_0 + (t-t_0)\mathbf{v}_f^* + \frac{1}{2}\mathbf{g}_0[(t-t_0) - 2(t_f-t_0)](t-t_0) \quad (17)$$

Therefore,

$$\mathbf{r}_k = \mathbf{r}_0 + (t_k - t_0)\mathbf{v}_f^* + \frac{1}{2}\mathbf{g}_0[(t_k-t_0) - 2(t_f-t_0)](t_k-t_0) \quad (18)$$

a) Three-point approximation

In the case of three-point approximation, there are two intervals $(N = 2)$. Therefore, based on the mentioned method, the algorithm for fixed-time interval is given by:

$$\mathbf{r}_1 = \mathbf{r}_0 + \frac{1}{2}(t_f - t_0)\mathbf{v}_f^* - \frac{3}{8}\mathbf{g}_0(t_f - t_0)^2 \quad (19)$$

$$\mathbf{r}_2 = \mathbf{r}_0 + (t_f - t_0)\mathbf{v}_f^* - \frac{1}{2}\mathbf{g}_0(t_f - t_0)^2 \quad (20)$$

$$\mathbf{g}_j = -\mu\mathbf{r}_j / r_j^3, \quad (j = 0,1,2) \quad (21)$$

$$\mathbf{v}_v^*(t_0) = \mathbf{v}_f^* - \frac{1}{4}(\mathbf{g}_0 + 2\mathbf{g}_1 + \mathbf{g}_2)(t_f - t_0) \quad (22)$$

where $r_j = |\,\mathbf{r}_j\,|$ and μ is the Earth gravitational constant. As can be seen in Eq. (21), the spherical Earth model is used here; however, elliptical Earth model can also be utilized in the formulation.

b) Four-point approximation

The algorithm for three-equal intervals is extended as follows:

$$\mathbf{r}_1 = \mathbf{r}_0 + \frac{1}{3}(t_f - t_0)\mathbf{v}_f^* - \frac{5}{18}\mathbf{g}_0(t_f - t_0)^2 \quad (23)$$

$$\mathbf{r}_2 = \mathbf{r}_0 + \frac{2}{3}(t_f - t_0)\mathbf{v}_f^* - \frac{4}{9}\mathbf{g}_0(t_f - t_0)^2 \quad (24)$$

$$\mathbf{r}_3 = \mathbf{r}_0 + (t_f - t_0)\mathbf{v}_f^* - \frac{1}{2}\mathbf{g}_0(t_f - t_0)^2 \quad (25)$$

$$\mathbf{g}_j = -\mu\mathbf{r}_j / r_j^3, \quad (j = 0,1,2,3) \quad (26)$$

$$\mathbf{v}_v^*(t_0) = \mathbf{v}_f^* - \frac{1}{6}(\mathbf{g}_0 + 2\mathbf{g}_1 + 2\mathbf{g}_2 + \mathbf{g}_3)(t_f - t_0) \quad (27)$$

6 Sensitivity Matrix with Velocity Constraint

The sensitivity matrix with final velocity constraint is defined as $Q_v(t) = \partial\mathbf{v}_v^*(t) / \partial\mathbf{r}(t)$. Here, we assume that t_0 is the current time, so we calculate $Q_v = \partial\mathbf{v}_v^*(t_0) / \partial\mathbf{r}_0$. By taking the partial derivative of Eq. (14) with respect to position $\mathbf{r}_0$ we have $(N \geq 2)$

$$\frac{\partial\mathbf{v}_v^*(t_0)}{\partial\mathbf{r}_0} = -\frac{1}{2}\left(T_1\frac{\partial\mathbf{g}_0}{\partial\mathbf{r}_0} + T_N\frac{\partial\mathbf{g}_N}{\partial\mathbf{r}_0}\right) - \frac{1}{2}\sum_{j=1}^{N-1}(T_j + T_{j+1})\frac{\partial\mathbf{g}_j}{\partial\mathbf{r}_0} \quad (28)$$

Using the relation of

$$\frac{\partial\mathbf{g}_j}{\partial\mathbf{r}_0} = -\frac{\mu}{r_j^3}\left[I - 3\mathbf{u}_{rj}\mathbf{u}_{rj}^T\right]\frac{\partial\mathbf{r}_j}{\partial\mathbf{r}_0} \quad (29)$$

one can obtain:

$$2Q_v = T_1 F_0 + T_N F_N \frac{\partial \mathbf{r}_N}{\partial \mathbf{r}_0} + \sum_{j=1}^{N-1} (T_j + T_{j+1}) F_j \frac{\partial \mathbf{r}_j}{\partial \mathbf{r}_0} \tag{30}$$

where I is a 3×3 identity matrix, $\mathbf{u}_{rj} = \mathbf{r}_j / r_j$, and

$$F_j = \frac{\mu}{r_j^3} \left[I - 3\mathbf{u}_{rj} \mathbf{u}_{rj}^T \right] \tag{31}$$

Therefore,

$$\frac{\partial \mathbf{g}_j}{\partial \mathbf{r}_0} = -F_j \frac{\partial \mathbf{r}_j}{\partial \mathbf{r}_0} \tag{32}$$

In the case that we utilize Eq. (18) for approximation of $\mathbf{r}_k$, we have

$$\frac{\partial \mathbf{r}_k}{\partial \mathbf{r}_0} = I - \frac{1}{2}(t_k - t_0)[(t_k - t_0) - 2(t_f - t_0)] F_0 \tag{33}$$

Hence, substitution for $\partial \mathbf{r}_k / \partial \mathbf{r}_0$ from the preceding relation into Eq. (30) results in:

$$\begin{aligned} 2Q_v = {} & T_1 F_0 + T_N F_N + \frac{1}{2} T_N F_N F_0 (t_f - t_0)^2 + \sum_{j=1}^{N-1} (T_j + T_{j+1}) F_j \\ & - \frac{1}{2} \sum_{j=1}^{N-1} (T_j + T_{j+1})(t_j - t_0)[(t_j - t_0) - 2(t_f - t_0)] F_j F_0 \end{aligned} \tag{34}$$

Special Case: Fixed-time interval

In the case of fixed-time interval, Eq. (34) simplifies to $(N \geq 2)$:

$$\begin{aligned} (2N) Q_v = {} & H_0 + H_N + \frac{1}{2}(t_f - t_0) H_N H_0 + 2\sum_{j=1}^{N-1} H_j \\ & + \frac{1}{N^2}(t_f - t_0) \sum_{j=1}^{N-1} (2N - j) j H_j H_0 \end{aligned} \tag{35}$$

where

$$H_j = \frac{\mu (t_f - t_0)}{r_j^3} \left[I - 3\mathbf{u}_{rj} \mathbf{u}_{rj}^T \right] \tag{36}$$

The solution is now given for several values of N as follows:

a) Two-point approximation $(N = 1)$:

$$2Q_v = H_0 + H_1 + \frac{1}{2}(t_f - t_0) H_1 H_0 \tag{37}$$

b) Three-point approximation $(N = 2)$:

$$4Q_v = H_0 + 2H_1 + H_2 + \frac{1}{2}(t_f - t_0) H_2 H_0 + \frac{3}{4}(t_f - t_0) H_1 H_0 \tag{38}$$

c) Four-point approximation $(N = 3)$:

$$\begin{aligned} 6Q_v = {} & H_0 + 2H_1 + 2H_2 + H_3 + \frac{1}{2}(t_f - t_0) H_3 H_0 \\ & + \frac{5}{9}(t_f - t_0) H_1 H_0 + \frac{8}{9}(t_f - t_0) H_2 H_1 \end{aligned} \tag{39}$$

7 Simulation Results

First, the obtained approximate solutions of the required velocity are compared to each other and to the exact numerical solution using the nonlinear flight simulation for the spherical-Earth model. The initial position of the vehicle is given as $[0\ 0\ R_e]^T$; where R_e is the Earth radius. The desired final velocity vector is $[2\ \ 3\ \ 0.5]^T$ km/s in the Earth-Centered Inertial (ECI) coordinates. The required velocity of the vehicle at its initial position with the fixed total flight time of 300 s is $\mathbf{v}_v^* = [2118.7\ \ 3178.1\ \ 3142.6]^T$ m/s, obtained using nonlinear flight simulation. The approximate solutions of the required velocity are also obtained using two-, three-, and four-point approximations and the results are listed in Table 1. As can be seen, three-point approximation has better accuracy than two-point approximation, but its results are relatively similar to the four-point approximation. Since, the midpoint positions have been obtained using constant gravity assumption, increasing the number of midpoints from 3, would not necessarily result in higher accuracy. To achieve a better result, the accuracy of the midpoint positions must be increased and that could be obtained by the present method.

Table 1. Calculation of required velocity for $t_f = 300$ s.

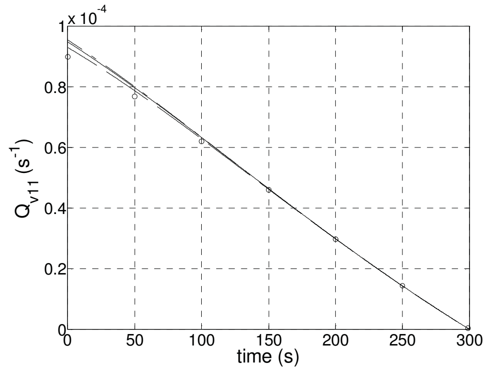
Method	$\mathbf{v}_v^*$ (m/s)
Two-Point	$[2103.1\ \ 3154.6\ \ 3174.2]^T$
Three-Point	$[2109.0\ \ 3163.4\ \ 3131.5]^T$
Four-Point	$[2110.1\ \ 3165.1\ \ 3123.5]^T$
Exact Solution	$[2118.7\ \ 3178.1\ \ 3142.6]^T$

Next, the approximate solutions for sensitivity matrix with velocity constraint are evaluated. Here, a vertical planar motion in the spherical-Earth model is considered. The vehicle nominal trajectory is given by

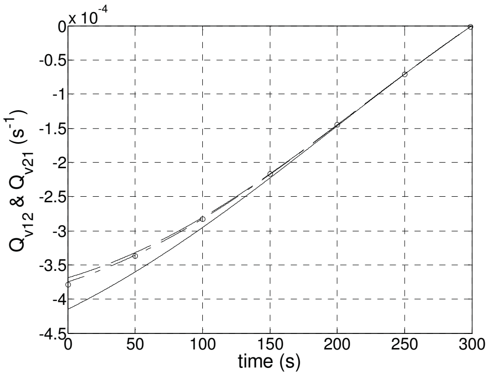
$$x_I = R_e \cos(\pi / 3) + 4.73 t^2 \tag{40a}$$

$$z_I = R_e \sin(\pi / 3) + 8.87 t^2 \tag{40b}$$

where (x_I, z_I) are the ECI coordinates.



(a)



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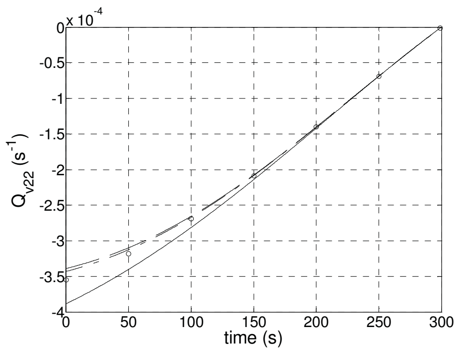
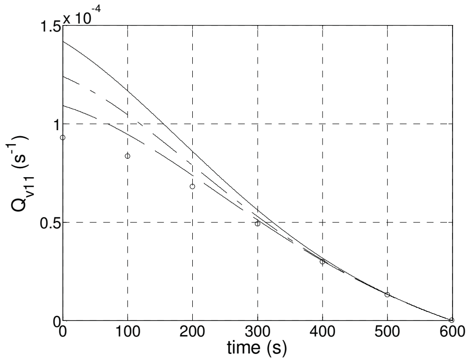
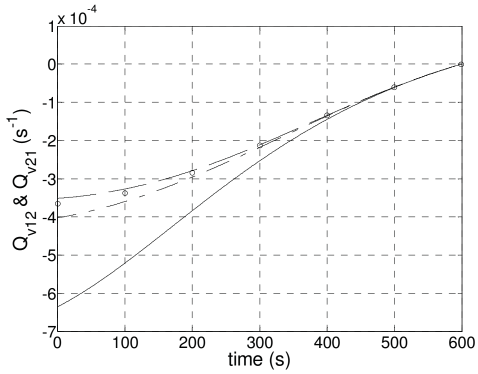


(c)

Figure 2. Different solutions of Q_v elements for $t_f = 300$ s; solid-line: 2-point approximation, dash-dotted line: 3-point approximation, dashed-line: 4-point approximation, and circles are the numerical solution using flight simulation.



(a)



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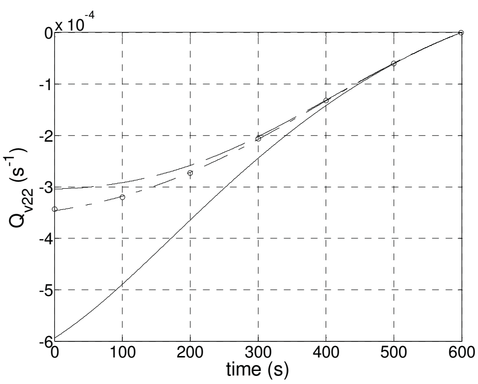

(c)

Figure 3. Different solutions of Q_v elements for $t_f = 300$ s (line representations are similar to Fig. 2).

Figures 2 and 3 depict the three different approximate solutions for Q_v elements at $t_f = 300$ and 600 s, respectively. In these figures, the two-point approximate solution $(N = 1)$, the three-point approximation $(N = 2)$, and four-point approximation $(N = 3)$ are shown by solid lines, dash-dotted lines, and dashed lines, respectively. In these figures, the numerical solution of $\Delta \mathbf{v}_v^* / \Delta \mathbf{r}$ using a nonlinear flight simulation is represented by circles. As shown, the accuracies of the three- and four-point solutions are higher than two-point solution; however, all the solutions have adequate accuracy for $t_f \leq 300$ s. For example, the two-point solution does not follow the curvature of the exact solution as seen in Figures 3b and c for $t_f = 300$ s. The accuracy of the four-point solution is relatively similar to that of the three-point solution, the reason for which has been explained earlier. The results are case-dependent and approximate solutions need to be modified for longer total flight times. The accuracy can also be enhanced using correction factors as treated in Ref. [14] for the two-point approximation. However, for longer total flight time, the accuracy of the positions of midpoints may also be increased and that could be obtained by the present method.

As a future study, the exact analytical solution for matrix Q_v, such as the one developed by Martin [4] for sensitivity matrix with final position vector, is suggested to be carried out. Also, perturbation techniques which may give accurate approximation for onboard calculations can be studied. As a comparative work, the methods of calculations of Q_v and their accuracies and computational burden can be compared.

8 Conclusions

This work presented an analytical solution for sensitivity matrix using piecewise linearization of gravitational acceleration when the total flight time is fixed. The method has been utilized to derive an approximate formula for sensitivity matrix with final velocity constraint. The accuracy of the method can be enhanced by increasing the number of midway points, which in turn, increases the computational burden. Therefore, there is a trade-off between accuracy and onboard computational burden. The method can be applied to elliptical Earth model, and also to three- or n-body problems.

Appendix: Sensitivity Matrix with Position Constraint

In this appendix, it is shown that the piecewise linear method can be applied to the sensitivity matrix with final position constraint. The sensitivity matrix with final position constraint was derived by Fred Martin in his thesis [4]; however, his derivation had some errors as stated by Battin on page 558 of Ref. [3] where he has added that "*The corrected version is courtesy of William M. Robertson of the Charles Stark Draper Laboratory.*"

Using Eq. (1b) we can write

$$\mathbf{r}(t_f) = \mathbf{r}_0 + (t_f - t_0)\mathbf{v}_0 + \sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} (t_f - t)\,\mathbf{g}(\mathbf{r}(t))\,dt \quad (41)$$

The spacecraft velocity that causes the desired final position $\mathbf{r}_f^*$ is referred to as the required velocity with final position constraint and is denoted by $\mathbf{v}_p^*$. Using the preceding relation we have:

$$\mathbf{v}_p^*(t_0) = \frac{\mathbf{r}_f^* - \mathbf{r}_0}{t_f - t_0} - \frac{1}{t_f - t_0}\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} (t_f - t)\,\mathbf{g}(\mathbf{r}(t))\,dt \quad (42)$$

Piecewise linear assumption for gravity, i.e. Eq. (3), gives:

$$\begin{aligned} 6(t_f - t_0)\mathbf{v}_p^*(t_0) &= 6(\mathbf{r}_f^* - \mathbf{r}_0) - T_1[3(t_f - t_0) - T_1]\mathbf{g}_0 \\ &-T_N^2\mathbf{g}_f - \sum_{j=1}^{N-1}[T_j^2 - T_{j+1}^2 + 3(T_j + T_{j+1})(t_f - t_j)]\mathbf{g}_j \end{aligned} \quad (43)$$

Partial differentiation with respect to position vector results in:

$$\begin{aligned} 6(t_f - t_0)\frac{\partial \mathbf{v}_p^*(t_0)}{\partial \mathbf{r}_0} &= -6I + T_1[3(t_f - t_0) - T_1]F_0 \\ &+ \sum_{j=1}^{N-1}[T_j^2 - T_{j+1}^2 + 3(T_j + T_{j+1})(t_f - t_j)]F_j\frac{\partial \mathbf{r}_j}{\partial \mathbf{r}_0} \end{aligned} \quad (44)$$

where F_j has been defined in Eq. (31).

The solutions are simplified for fixed-time interval, $T = (t_f - t_0)/N$, as follows:

$$\mathbf{v}_p^*(t_0) = \frac{\mathbf{r}_f^* - \mathbf{r}_0}{t_f - t_0} - \frac{t_f - t_0}{6N^2}\left\{(3N-1)\mathbf{g}_0 + \mathbf{g}_f + 6\sum_{j=1}^{N-1}(N-j)\,\mathbf{g}_j\right\} \quad (45)$$

$$\frac{\partial \mathbf{v}_p^*(t_0)}{\partial \mathbf{r}_0} = \frac{-I}{t_f - t_0} + \frac{3N-1}{6N^2}H_0 + \frac{1}{N^2}\sum_{j=1}^{N-1}(N-j)H_j\frac{\partial \mathbf{r}_j}{\partial \mathbf{r}_0} \quad (46)$$

where $H_i = (t_f - t_0)F_i$.

Depending on the desired accuracy, an appropriate approximation must be selected for $\partial \mathbf{r}_j / \partial \mathbf{r}_0$. For example, using Eq. (7) for $(N = 2)$ we obtain:

$$\begin{aligned} \mathbf{r}_k &= \mathbf{r}_0 + (t_k - t_0)\mathbf{v}_p^* + \frac{(t_k - t_0)^2}{6(t_f - t_0)}\Big\{[3(t_f - t_k) \\ &+2(t_k - t_0)]\mathbf{g}_0 + (t_k - t_0)\mathbf{g}_f\Big\} \end{aligned} \quad (47)$$

$$\frac{\partial \mathbf{r}_k}{\partial \mathbf{r}_0} = I + \frac{k}{N}(t_f - t_0)\frac{\partial \mathbf{v}_p^*(t_0)}{\partial \mathbf{r}_0} - \frac{k^2}{6N^3}(t_f - t_0)(3N-k)H_0 \quad (48)$$

Substitution of Eq. (48) in Eq. (46) and rearrangement results in:

$$\frac{\partial \mathbf{v}_p^*(t_0)}{\partial \mathbf{r}_0} = \left[I - \frac{t_f - t_0}{N^3}\sum_{j=1}^{N-1} j(N-j)H_j\right]^{-1} J \quad (49)$$

where

$$J = -\frac{1}{t_f - t_0} I + \frac{3N-1}{6N^2} H_0 + \frac{1}{N^2} \sum_{j=1}^{N-1} (N-j) H_j \quad (50)$$
$$+ \frac{t_f - t_0}{6N^5} \sum_{j=1}^{N-1} (N-j)(3N-j) j^2 H_j H_0$$

The sensitivity matrix $\partial \mathbf{v}_p^*(t_0) / \partial \mathbf{r}_0$ in the right-hand side of Eq. (48) may also be replaced by a three or more-point approximation in order to obtain a more simplified relation.

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