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Determination of the Best-Fitting Reference Orbit for a LEO Satellite Using the Lagrange Coefficients

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Linearization of the nonlinear equations and iterative solution is the most well-known scheme in many engineering problems. For geodetic applications of the LEO satellites, e.g. the Earth's gravity field recovery, one needs to provide an initial guess of the satellite location or the so-called reference orbit. Numerical integration can be utilized to generate the reference orbit if a satellite's state vector, i.e. position and velocity, is known at a reference epoch. However, the numerically integrated orbit deviated from the real orbit due to imperfect force models. The more accurate the reference orbit, the less linearization error occurs. The deviation between the reference and real orbit can be minimized using the least squares method. Different analytical and numerical techniques have been developed for calculation of the design matrix of the least squares method. Herein, we have generalized the idea of the Lagrange coefficients for determination of the design matrix's entries in the gravitational field of an attracting inhomogeneous mass body. Numerical implementation of the proposed method shows its high performance.

INTRODUCTION

The reference orbit is an initial approximation of the observed satellite orbit that can be used to linearize purposes. It is needed for many applications in satellite geodesy, *e.g.* reduced dynamic orbit determination by Kalman filtering and the Earth's gravity field recovery using the space-borne measurements [1]. The reference orbit of a satellite could be used as a trend of the state vectors, observed for example, from Global Positioning System (GPS). Therefore, any observation of the satellites could be divided into two parts: the trend or reference component (model- derived) and the remaining part (residual). The residuals increase linearly in time even if it is zero at the initial time, *i.e.* when the reference orbit coincides with the real orbit. It happens because of diversity between actual force field acting on a satellite and the force models utilized for the reference orbit computation [2]. A closer reference orbit to the actual orbit could be achieved by integration (dynamic method) using an already known geopotential model, *e.g.* EGM96, EIGENs and GEMs [3]. The result of integration, *i.e.* the reference orbit, is highly dependent on the selection of initial position of satellite. The resulting reference orbit will be non-realistic if the initial values are not selected properly. For the proper selection of them, the State Transition Matrix (STM) at the initial epoch is computed by the Lagrange method.

In Sconzo [4], the expansion series of Lagrange coefficients was symbolically determined in a central field using FORMAC computer program. Montenbruck [5] developed the mathematical bases for the solution of an ordinary differential equation based on the expansion of the unknown vector in the Taylor series. Basically, he introduced a new method for numerical orbit integration with a given initial state vector. Bem and Szczodrowska-Kozar[6] derived expansion coefficients of the Lagrange coefficients up to degree 20. The coefficients were used for orbit determination of the

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solar system planets. They determined the coefficients for the two- and three-body problems with the assumption of central field for the attracting bodies. The idea was implemented for dynamic orbit determination of the Low Earth orbit (LEO) satellites using the GPS-based observations in Feng [7]. The Lagrange coefficients and their time derivatives were employed for calculation of the STM in dynamic processing for orbit improvement. Laplace's method of initial orbit determination with the angular observations based on the Lagrange coefficients was developed in Lin and Wang [8].

In this article, the Lagrange coefficients are expanded from the central field to the zonal Earth's gravity field.

THE BEST FITTING REFERENCE ORBIT

The reference orbit deviates from the real orbit for two reasons: First, using erroneous initial values (conditions), and second, the imperfect modeling of the forces acting on a satellite. Consequently, the reference positions of satellites, derived from the reference orbit, are different from the actual positions. The differences in positions are called location error. In order to minimize the location error, the reference orbit should be computed as close as possible to the real orbit [1]. In this paper, we propose the least squares approach for selecting the initial conditions in a way that the total misfit of the reference and observed orbit is minimized.

When integrating a reference orbit in a time interval, the location error is zero at the initial time and gradually increases to a maximum at the end of time interval, *i.e.* the v-shaped pattern of the two orbits differences. It would be better to uniformly distribute the differences over the interval. In other words, the v-shaped pattern of the differences is changed in such a way that the maximum deviation of the two orbits should be minimized to a great extent. This orbit is usually called the best-fitting reference orbit in the sense of the least squares of orbit deviations. An orbit could be described by a dynamic process (a linear differential equation of the first order) as:

$$\dot{\underline{s}}^{ref}(t) = \underline{f}^{ref}(t, \underline{s}^{ref}(t)), \qquad \underline{s}^{ref}(t_0) = \underline{s}_0^{ref} \tag{1}$$

In the current formulation, instead of using $\underline{s}_0$ as the initial values, the initial values corresponding to the reference orbit ($\underline{s}_0^{ref}$) are used. The question is how we can suitably determine $\underline{s}_0^{ref}$ which results in a uniform deviation. The idea of the best fitting reference orbit in the sense of minimum least squares differences has been used by Ballani [9]. Following this idea, we assume the given initial state vector as an approximate quantity and try to find the vector correction $\underline{ds}$:

$$\underline{s}_0^{ref} = \underline{s}_0 + \underline{ds} \tag{2}$$

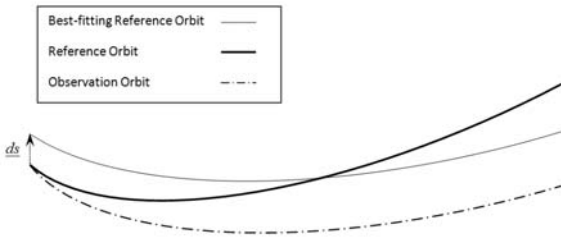


Figure 1. Observation Orbit, Reference Orbit and Best-fitting Reference Orbit.

The problem is now to find an appropriate correction to the initial estimate of the state vector in a way that the deviation of the reference orbit from the observed orbit is minimized. This can be formulated as:

$$\begin{cases} \underline{s}_i + \Delta\underline{s}_i = \underline{s}^{ref}(t_i, \underline{s}_0^{ref}) \text{ for } i = 0, 1, \cdots, N \\ \sum\limits_{i=1}^{N} \|\Delta\underline{s}_i\|^2 \rightarrow \min \end{cases} \tag{3}$$

where $\underline{s}_i$ is observation vector, $\Delta\underline{s}_i$ is the difference between observations and the reference orbit. $\underline{s}^{ref}(t_i, \underline{s}_0^{ref})$ represents N (total number of the integration points) non-linear equations in terms of $\underline{s}_0^{ref}$ or equivalently $\underline{ds}$. Assume the sought-after correction is small enough for the linearization to yield accurate approximation of the equations.

$$\underline{s}^{ref}(t_i, \underline{s}_0^{ref}) = \underline{s}^{ref}(t_i, \underline{s}_0) + \frac{\partial \underline{s}_i^{ref}}{\partial \underline{s}_0^{ref}} \underline{ds} \tag{4}$$

Inserting the linearized form of the state vector into Eq. (3), it is recast into:

$$\begin{cases} \Delta\underline{s}_i = \frac{\partial \underline{s}_i^{ref}}{\partial \underline{s}_0^{ref}} \underline{ds} - \left(\underline{s}_i - \underline{s}^{ref}(t_i, \underline{s}_0)\right) \\ \sum\limits_{i=1}^{N} \|\Delta\underline{s}_i\|^2 \rightarrow \min \end{cases} \tag{5}$$

or equivalently,

$$\begin{cases} \underline{d} = \underline{A}\,\underline{ds} - \underline{d\ell} \\ \sum\limits_{j=1}^{N} \|\underline{d}\| \rightarrow \min \end{cases} \tag{6}$$

with the misfit vector $\underline{d} = [\Delta\underline{s}_i]_{6N\times 1}$, $\|\underline{d}\|$ the norm of the misfit vector, the design matrix $\underline{A}$ and the observation misclosure vector $\underline{dl} = \left[\underline{s}_i - \underline{s}_{i0}^{ref}\right]_{6N\times 1}$. Applying the method of least squares yields:

$$\underline{d\hat{s}} = (\underline{A}^T \underline{P}\, \underline{A})^{-1} \underline{A}^T \underline{P}\, \underline{dl} \tag{7}$$

where, $\underline{P}$ is the weight matrix of the observations. For ease of implementation, it is set equal to the identity matrix. Except the design matrix $\underline{A}$, the remaining vectors and matrices can easily be computed.

As defined, the design matrix entries are the partial derivatives of the state vectors with respect to the initial state vector which is called the transition state vector. The design matrix $\underline{A}$ is:

$$\underline{A} = \begin{bmatrix} \Phi(t_0, t_0) \\ \Phi(t_1, t_0) \\ \vdots \\ \Phi(t_N, t_0) \end{bmatrix} \tag{8}$$

$\Phi(t_i, t_0)$ is the state transition matrix which transfers state vector from initial time, t_0, to any times.

There are different approaches for computation of the state transition matrix. Herein, we introduce the Lagrange method to compute the state transition matrix.

MATHEMATICAL MODELING OF LAGRANGE COEFFICIENTS

Motion of a satellite in a gravitational field of an attractive body is expressed by the equations of motion [10]:

$$\ddot{\underline{r}} = \nabla V, \qquad \underline{r}(t_0) = \underline{r}_0, \qquad \dot{\underline{r}}(t_0) = \dot{\underline{r}}_0 \tag{9}$$

where V stands for the gravitational potential of the attractive body acting on the satellite.

Equation (9) is the equation of motion which plays the fundamental role for orbit determination.

In general, the solution can be expressed in terms of the initial position and velocity vectors. The coefficients of this linear combination are called Lagrange coefficients:

$$\begin{cases} \underline{r}(t) = f(t)\underline{r}(t_0) + g(t)\dot{\underline{r}}(t_0) \\ \dot{\underline{r}}(t) = \dot{f}(t)\underline{r}(t_0) + \dot{g}(t)\dot{\underline{r}}(t_0) \end{cases} \tag{10}$$

where $f, g, \dot{f}, \dot{g}$ are the time-dependent transfer coefficients. The solution of orbit determination problem is equivalent to the determination of the coefficients $f, g, \dot{f}$ and $\dot{g}$ [12].

In the two-body problem, *i.e.* the simplest form of the orbital motion, the coefficients are scalar functions [11]. However, Eq. (10) should be reformulated for the orbital motion in a non-Keplerian gravitational field. The general form can be obtained by substitution of the scalar coefficient functions by a matrix form as follows:

$$\begin{cases} \underline{r}(t) = \underline{F}(t)\underline{r}(t_0) + \underline{G}(t)\dot{\underline{r}}(t_0) \\ \dot{\underline{r}}(t) = \dot{\underline{F}}(t)\underline{r}(t_0) + \dot{\underline{G}}(t)\dot{\underline{r}}(t_0) \end{cases} \tag{11}$$

where $\underline{F}(t)$, $\underline{G}(t)$, $\dot{\underline{F}}(t)$ and $\dot{\underline{G}}(t)$ are matrix-valued functions:

$$\underline{F} = \begin{bmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{bmatrix}, \ \dot{\underline{F}} = \begin{bmatrix} \dot{f}_1 & 0 & 0 \\ 0 & \dot{f}_2 & 0 \\ 0 & 0 & \dot{f}_3 \end{bmatrix} \tag{12a}$$

and

$$\underline{G} = \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix}, \ \dot{\underline{G}} = \begin{bmatrix} \dot{g}_1 & 0 & 0 \\ 0 & \dot{g}_2 & 0 \\ 0 & 0 & \dot{g}_3 \end{bmatrix} \tag{12b}$$

The state transition matrix is defined if the Lagrange coefficients of the position and velocity vectors expansion are known. To derive the coefficients, the Taylor expansion of the coefficients in terms of the polynomial expressions around the initial epoch is used:

$$f_i(t) = \sum_{n=0}^{\infty} \frac{1}{n!} f_i^{(n)} \mid_{t=t_0} (t - t_0)^n$$

$$g_i(t) = \sum_{n=0}^{\infty} \frac{1}{n!} g_i^{(n)} \mid_{t=t_0} (t - t_0)^n, \ \ i = 1, 2, 3, \tag{13}$$

and their time-derivatives are:

$$\dot{f}_i = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} f_i^{(n)} \mid_{t=t_0} (t - t_0)^{n-1}$$

$$\dot{g}_i = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} g_i^{(n)} \mid_{t=t_0} (t - t_0)^{n-1}, \ \ i = 1, 2, 3 \tag{14}$$

Similarly, one can expand the position vector in terms of the time derivatives of the position vector:

$$\underline{r}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \underline{r}(t)}{\partial t^n}\bigg|_{t=t_0} (t - t_0)^n \tag{15}$$

Taking time derivatives of the Eq. (11) gives the time derivatives of the position vector in terms of the time derivatives of Lagrange coefficients:

$$\underline{r}(t) = \sum \frac{1}{n!} \left(\underline{F}^{(n)}(t)\underline{r}_0 + \underline{G}^{(n)}(t)\dot{\underline{r}} \right)\Big|_{t=t_0} (t - t_0)^n \tag{16}$$

where

$$\underline{F}^{(n)} = \begin{bmatrix} f_1^{(n)} & 0 & 0 \\ 0 & f_2^{(n)} & 0 \\ 0 & 0 & f_3^{(n)} \end{bmatrix}, \ \underline{G}^{(n)} = \begin{bmatrix} g_1^{(n)} & 0 & 0 \\ 0 & g_2^{(n)} & 0 \\ 0 & 0 & g_3^{(n)} \end{bmatrix} \tag{17}$$

Eq. (16) is used to compute $\underline{F}^{(n)}$ and $\underline{G}^{(n)}$. In this paper, the coefficients are derived in the zonal gravitational field as a representative example.

THE LAGRANGE COEFFICIENTS FOR THE ZONAL FIELD

The higher-order terms of the Earth's gravitational potential are the most important perturbing sources for the Low Earth Orbiter (LEO) satellites besides

the perturbation effect of the Earth's flattening [13]. The zonal coefficients of the Earth's gravitational field ($C_{n,0}$) can cause secular perturbations in the LEO satellites orbit. The sectorial and tesseral spherical harmonics affect the motion in a periodic manner with much smaller influences [10]. Therefore, it is necessary to count the perturbation caused by the zonal terms for achieving a reference orbit with a reasonable accuracy.

The zonal gravitational potential terms as a function of the curvilinear coordinates r, φ is:

$$V(r,\varphi) = \frac{GM}{r}[1 + \sum_{n=2}^{\infty} (\frac{a_e}{r})^n \bar{C}_{n0} \bar{P}_{n0}(\sin\varphi)] \tag{18}$$

where GM is the product of gravitational constant and the Earth's mass, a_e is the Earth's equatorial radius, $\overline{C}_{n0}$ is degree n zonal coefficients of the Earth's gravity field and $\overline{P}_{n0}(sin\varphi)$ are the associated Legendre functions of degree n and order zero.

As shown in Eq. (16), we should compute the nth time derivative of $\underline{r}$ for computation of Lagrange coefficients. The zero and first order derivatives of $\underline{r}$ are given as the initial value and we need to start from the second order derivatives on.

Due to the Earth's spherical shape, the representation of gravitational acceleration in curvilinear coordinates (r, φ) is more appropriate. However, the equation of motion (Eq. 9) is given in the Cartesian coordinates. Therefore, the partial derivatives of the Earth's gravitational potential are taken with respect to (r, φ) and the results are transferred into the Cartesian coordinate:

$$\underline{r}^{(2)}(x,y,z) = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = J_{r\varphi}^{xyz} \begin{bmatrix} \frac{\partial V}{\partial r} \\ \frac{\partial V}{\partial \varphi} \end{bmatrix} \tag{19}$$

where $J_{r\varphi}^{xyz}$ is the Jacobian matrix of Cartesian coordinates towards the curvilinear ones. Using the mathematical relationship of these two sets of coordinates, one can derive:

$$\underline{r}^{(2)} = (a-b)\underline{r} + ez\hat{k} \tag{20}$$

where $\hat{k} = [0,0,1]$ is unit vector in direction Z-axis, and a, b and e are scalar coefficients.

The first and second terms in the right hand side of Eq. (20) show the radial and non-radial components of the gravitational acceleration. The radial term is the projection of the acceleration vector onto $\underline{r}$ and non-radial term is the remaining part of acceleration.

The unknown coefficients of Eq. (20) are expressed in terms of position and the partial derivatives of gravitational potential in terms of the curvilinear coordinate.

$$\begin{aligned} a &= \frac{1}{r} V_r \\ b &= (\frac{z}{r^2\sqrt{x^2+y^2}}) V_\varphi \\ e &= \frac{1}{z\sqrt{x^2+y^2}} V_\varphi \end{aligned} \tag{21}$$

Similarly, the third-order time derivative of $\underline{r}$ is:

$$\underline{r}^{(3)} = (\dot{a}-\dot{b})\underline{r} + (a-b)\dot{\underline{r}} + \dot{e}z\hat{k} + e\dot{z}\hat{k} \tag{22}$$

and the same is true for $\underline{r}^{(4)}$:

$$\underline{r}^{(4)} = (\ddot{a}-\ddot{b})\underline{r} + 2(\dot{a}-\dot{b})\dot{\underline{r}} + (a-b)\ddot{\underline{r}} + \ddot{e}z\hat{k} + 2\dot{e}\dot{z}\hat{k} + e\ddot{z}\hat{k} \tag{23}$$

By substituting Eq. (20) into Eq. (23):

$$\begin{aligned} \underline{r}^{(4)} &= [(\ddot{a}-\ddot{b}) + (a-b)^2]\underline{r} \\ &+ 2(\dot{a}-\dot{b})\dot{\underline{r}} + [\ddot{e} + 2e(a-b) + e^2]z\hat{k} + 2\dot{e}\dot{z}\hat{k} \end{aligned} \tag{24}$$

The Lagrange coefficients for the full field up to the fourth order are presented in Table (1).

Assuming that $f_i^{(5)}$ and $g_i^{(5)}$ are equal to the Keplerian coefficients (please see appendix C for more details) instead of $f_i^{(5)} = g_i^{(5)} = 0$ improves the obtained accuracy [14].

For predicting state vector, we use

$$s_i(t) = L_{i,k} s_k(t_0) \tag{25}$$

In a matrix form, it reads:

$$\underline{s}(t) = \underline{L}(t_0, \underline{s}_0)\underline{s}(t_0) \tag{26}$$

Table 1. Taylor expansion coefficients in zonal field.

n	$f_1^{(n)} = f_2^{(n)}$	$g_1^{(n)} = g_2^{(n)}$	$f_3^{(n)}$	$g_3^{(n)}$
0	1	0	1	0
1	0	1	0	1
2	$(a-b)$	0	$(a-b)+h$	0
3	$(\dot{a}-\dot{b})$	$(a-b)$	$(\dot{a}-\dot{b})+\dot{e}$	$(a-b)+e$
4	$(\ddot{a}-\ddot{b})+(a-b)^2$	$2(\dot{a}-\dot{b})$	$(\ddot{a}-\ddot{b})+(a-b)^2+e^2+2(a-b)e+\ddot{e}$	$2(\dot{a}-\dot{b})+2\dot{e}$

where

$$\underline{L} = \begin{bmatrix} f_1 & 0 & 0 & g_1 & 0 & 0 \\ 0 & f_2 & 0 & 0 & g_2 & 0 \\ 0 & 0 & f_3 & 0 & 0 & g_3 \\ \dot{f}_1 & 0 & 0 & \dot{g}_1 & 0 & 0 \\ 0 & \dot{f}_2 & 0 & 0 & \dot{g}_2 & 0 \\ 0 & 0 & \dot{f}_3 & 0 & 0 & \dot{g}_3 \end{bmatrix} \tag{27}$$

STATE TRANSITION MATRIX COMPUTATION

The state transition matrix maps deviations in the state vector from one epoch to another. In non-linear dynamic models, deviations in the state are mapped from initial epoch, t_0, to an arbitrary epoch, t. The state transition matrix can be obtained as follows [15]:

$$\Phi_{i,j} = \frac{\partial s_i(t)}{\partial s_j(t_0)} \tag{28}$$

or in a matrix form:

$$\underline{\Phi}(t,t_0) \equiv \frac{\partial \underline{s}(t)}{\partial \underline{s}_0} = \begin{bmatrix} \frac{\partial \underline{r}(t)}{\partial \underline{r}_0} & \frac{\partial \underline{r}(t)}{\partial \underline{\dot{r}}_0} \\ \frac{\partial \underline{\dot{r}}(t)}{\partial \underline{r}_0} & \frac{\partial \underline{\dot{r}}(t)}{\partial \underline{\dot{r}}_0} \end{bmatrix} \tag{29}$$

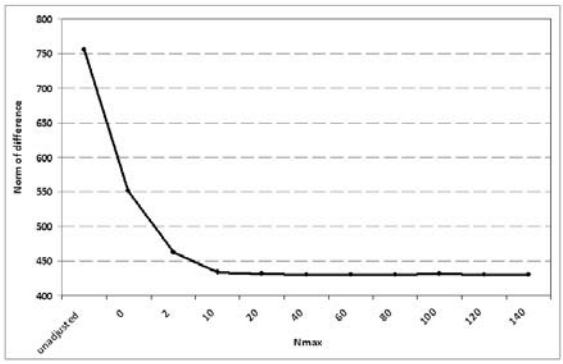


Figure 2. Norm of differences between the simulated real and reference orbits.

By substituting Eq. (25) into Eq. (28):

$$\Phi_{i,j} = \frac{\partial s_i(t)}{\partial s_j(t_0)} = \frac{\partial L_{i,k}}{\partial s_j(t_0)} s_k(t_0) + L_{i,k} \frac{\partial s_k(t_0)}{\partial s_j(t_0)} \tag{30}$$

where $\frac{\partial s_k(t_0)}{\partial s_j(t_0)}$ takes a very simple identity matrix form:

$$\frac{\partial s_k(t_0)}{\partial s_j(t_0)} = \begin{cases} 1 & k = j \\ 0 & k = j \end{cases} \tag{31}$$

Hence,

$$\Phi_{i,j} = \frac{\partial s_i(t)}{\partial s_j(t_0)} = \frac{\partial L_{i,k}}{\partial s_j(t_0)} s_k(t_0) + L_{i,j} \tag{32}$$

Equivalently, in matrix form, it reads:

$$\underline{\Phi}(t,t_0) = \sum_{k=1}^{6} \frac{\partial \underline{l}_k}{\partial \underline{s}(t_0)} s_k(t_0) + \underline{L} \tag{33}$$

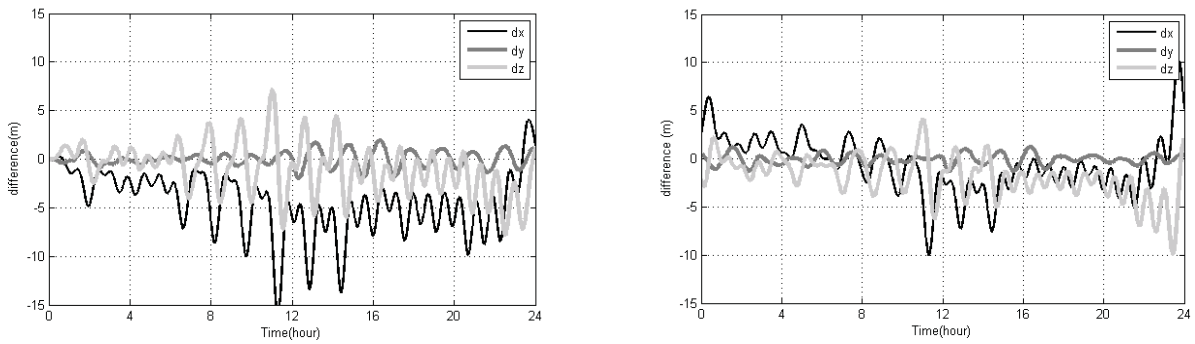
where $\underline{l}_k$ is the k-th column vector of the Lagrange matrix $\underline{L}$. To reduce the influence of truncation error of the Taylor expansion of f_i and g_i, $\underline{\Phi}(t_n,t_0)$ should be computed recursively:

$$\underline{\Phi}(t_n,t_0) = \underline{\Phi}(t_n,t_{n-1})\underline{\Phi}(t_{n-1},t_{n-2})...\underline{\Phi}(t_1,t_0) \tag{34}$$

NUMERICAL ANALYSIS

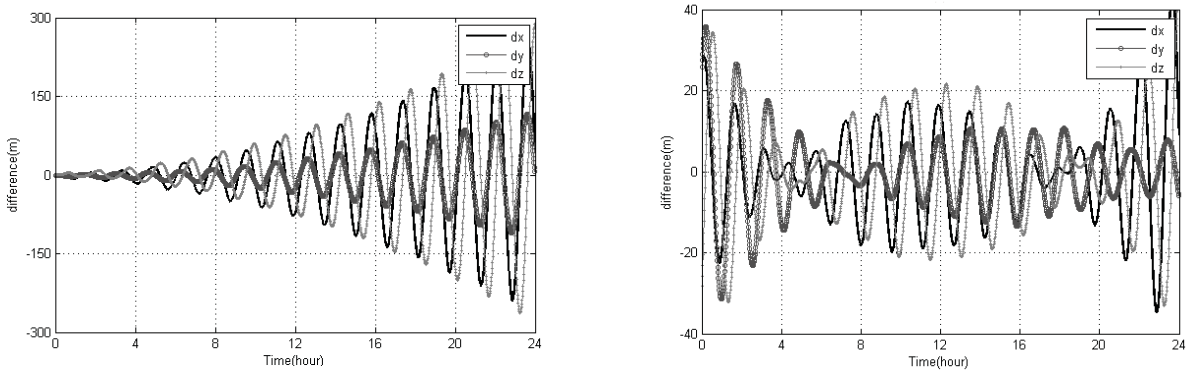
In this paper, both EGM96 and GGM02s [16] up to degree and order 140 were considered as the reference and the pseudo-real field for the CHAllenging Mini-Satellite Payload (CHAMP) orbit propagation [17] .

Using the Earth's gravitational field of different degrees leads to different state transition matrices. Consequently, one expects different reference orbit due to differences in the reference force field. The best-fitting reference orbit is the solution which has the



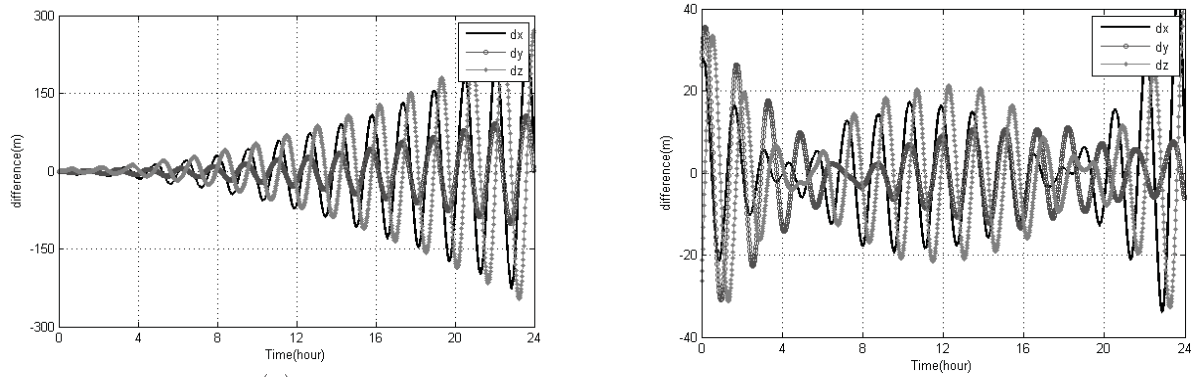
(a) (b)

Figure 3. Coordinate differences between the simulated real and reference orbits, a) Initial reference orbit, b) The best-fitting reference orbit.



(a) (b)

Figure 4. Coordinate differences between the GRACE-A real orbit and a) the initial reference orbit, b) the best-fitting reference orbit.



(a) (b)

Figure 5. Coordinate differences between the GRACE-B real orbit and a) the initial reference orbit, b) the best-fitting reference orbit.

minimum deviation with respect to the observed orbit in the sense of the sum of the squared distance.

Figure 2 shows the fact that increasing the maximum degree n to a specified degree reduces the sum of differences between the simulated observation and reference orbits. It remains constant after Nmax=20. This behavior is due to the fact that the correction vector ($\underline{ds}$) is too small compared to the state vector. Moreover, it proves that the effect of short wavelengths of the Earth's gravity field on the STM is attenuated at the satellite altitude.

Figure 3-a shows coordinate differences between the simulated real and the reference orbits which are obtained by orbit integration from epoch t_0. Figure 3-b shows coordinate differences between the simulated real and the best-fitting reference orbits obtained by the least square approach based on the transition matrix derived by Eq. (33).

To show the performance of the proposed methods in real data, it is employed for Precise Science Orbit (PSO) of the GRACE twin satellites released by GFZ [18]. The PSO minus the reference and best-fitting reference orbits for GRACE-A and GRACE-B are shown in Figures 4 and 5.

As seen, the difference of the real and best fitting reference orbits is nearly twice as much as that of the simulated observations. Amplification of the differences is due to neglecting the other source of perturbation forces (*e.g.* air drag, solid and ocean tides, solar radiation pressure, third body attraction,...) acting on satellite in real observations. Nevertheless, introducing the best fitting reference orbit solely based on the zonal gravitational field of the Earth reduces the differences by one order of magnitude.

CONCLUSION

In this article, we extend the idea of Lagrange coefficients from the central to the zonal gravitational field of the Earth or any massive attracting body. Moreover, we compute state transition matrix based on the extension required for determination of the best-fitting reference orbit. By adding the zonal terms, the norm of difference between the observed and reference

orbits decreases up to degree twenty. It remains for higher order zonal harmonics. This behavior is due to the fact that vector correction ($\underline{ds}$) is too small compared to the state vector. Moreover, it proves that the effect of short wavelengths of the Earth's gravity field on the STM is attenuated at the satellite altitude.

Numerical performance of the proposed method is checked both in simulated and real data. In the worst case, the best fitting reference orbit based on the Lagrange coefficients reduces the differences by one order of magnitude.

APPENDIX A

The first and second order derivative of a, b and e

$$a = \frac{1}{\sqrt{x^2+y^2+z^2}}u_r$$
$$b = (\frac{z}{(x^2+y^2+z^2)\sqrt{x^2+y^2}})u_\varphi$$
$$e = \frac{1}{z\sqrt{x^2+y^2}}u_\varphi \tag{35}$$

All of the scalars have the form of:

$$F(x,y,z,t) = G(x,y,z)\frac{\partial U(r,\varphi)}{\partial \alpha} \tag{36}$$

F and G are known functions and $\alpha \in [r,\varphi]$. The first and second order derivatives of F are:

$$\dot{F} = \frac{dG}{dt}\frac{\partial u}{\partial \alpha} + G\frac{d}{dt}(\frac{\partial u}{\partial \alpha})$$
$$\ddot{F} = \frac{d^2G}{dt^2}\frac{\partial u}{\partial \alpha} + 2\frac{dG}{dt}\frac{d}{dt}(\frac{\partial u}{\partial \alpha}) + G\frac{d^2}{dt^2}(\frac{\partial u}{\partial \alpha}) \tag{37}$$

where

$$\frac{dG}{dt} = \frac{\partial G}{\partial x_i}\dot{x}$$
$$\frac{d^2G}{dt^2} = \frac{\partial^2 G}{\partial x_i \partial x_j}\dot{x}_i\dot{x}_j + \frac{\partial G}{\partial x_i}\ddot{x}_i \tag{38}$$

The chain role should be used to compute the first- and second- order derivatives of $\frac{\partial u}{\partial \alpha}$ with respect to time too.

$$\frac{d}{dt}(\frac{\partial u}{\partial \alpha}) = \frac{\partial^2 u}{\partial \alpha \partial r_j}\dot{r}_j$$
$$\frac{d^2}{dt^2}(\frac{\partial u}{\partial \alpha}) = \frac{\partial^3 u}{\partial \alpha \partial r_j \partial r_k}\dot{r}_j\dot{r}_k + \frac{\partial^2 u}{\partial \alpha \partial r_j}\ddot{r}_j \tag{39}$$

For example:

$$\frac{du_\varphi}{dt} = u_{r\varphi}\dot{r} + u_{\varphi\varphi}\dot{\varphi}$$
$$\frac{d^2u_\varphi}{dt^2} = u_{r\varphi}\ddot{r} + u_{\varphi\varphi}\ddot{\varphi} + (u_{rr\varphi}\dot{r} + u_{r\varphi\varphi}\dot{\varphi})\dot{r} + (u_{r\varphi\varphi}\dot{r} + u_{\varphi\varphi\varphi}\dot{\varphi})\dot{\varphi} \tag{40}$$

APPENDIX B

The derivatives of circular coordinate

Like derivative of function G, the first order derivative of $r(r \in [r,\varphi])$ is:

$$\dot{r} = (\frac{\partial r}{\partial x_i})\dot{x}_i \tag{41}$$

and the second order derivative is:

$$\ddot{r} = (\frac{\partial^2 r}{\partial x_i \partial x_j})\dot{x}_i\dot{x}_j + (\frac{\partial r}{\partial x_i})\ddot{x}_i \tag{42}$$

APPENDIX C

The derivative of $\underline{L}$ respect to $\underline{s}_0$

For simplification, in order to compute $\frac{\partial L_{i,k}}{\partial s_j(t_0)}$ Eq. (32), we suppose that the Lagrange matrix ($\underline{L}$) is keplerian. The expansion coefficients $f_i^{(n)}$ and $g_i^{(n)}$ in two-body problem are given consequently in Table 2, where

$$r_0 = \|\underline{r}_0\|$$
$$v_0 = \|\dot{\underline{r}}\|$$
$$\sigma = \underline{r}_0 \cdot \dot{\underline{r}}_0 \tag{43}$$

As it is shown, the f_i, g_i, and the first derivations of them are functions of $\underline{r}_0$ and $\dot{\underline{r}}$. The computation of their derivatives towards $\underline{s}_0$ are not complicated.

Table 2. Lagrange coefficients for central field.

n	$f_1^{(n)} = f_2^{(n)} = f_3^{(n)}$	$g_1^{(n)} = g_2^{(n)} = g_3^{(n)}$
0	1	0
1	0	1
2	$\frac{-GM}{r_0^3}$	0
3	$3\frac{GM}{r_0^5}\sigma$	$\frac{-GM}{r_0^3}$
4	$\frac{GM}{r^5}(3v_0^2 - 2\frac{GM}{r_0} - 15\frac{\sigma^2}{r_0^2})$	$6\frac{GM}{r_0^5}\sigma$
5	$-\frac{15GM}{r_0^7}\sigma(3v_0^2 - 2\frac{GM}{r_0} - 7\frac{\sigma^2}{r_0^2})$	$\frac{GM}{r_0^5}(9v_0^2 - 8\frac{GM}{r_0} - 45\frac{\sigma^2}{r_0^2})$

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