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Analysis of a solid state wave gyroscope with a thin shell cylindrical resonator and calculation of its conversion factor

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In this work the equations of motion of a Solid State Wave Gyroscope (SWG) with rotary thin cylindrical shell resonator is analyzed using the shell and plates elasticity theory. The gyroscope conversion factor found in this analytical study corresponds with the experimental results obtained and listed in the References. The function of the SWG to measure the angular velocity or the rotating angle depends on the type of the resonator's excitation. This paper analyzes two types of excitations: a positional and a parametric excitation. The former has a rate gyroscope whereas the latter has a rate integrating gyroscope. The equations of motion are solved using Bobnov-Galerkin method, where the forces produced from excitation electrodes are assumed to be the external non-conservative forces. Finally, the relationship between the input and output of the gyroscope are derived to demonstrate that the type of excitation determines the type of gyroscope.

INTRODUCTION

The need for high precision sensitive elements in inertial navigation and automatic flight control systems has resulted in designing modern gyroscopic equipment. SWG is a collection of novel equipment based on the inertial properties of static waves, which become excited by rotating symmetrical shells. The effect of static waves in rotating objects with axial symmetry is discovered by Bryan as early as 1890 and subsequent experiments have confirmed his idea [2].

By definition, SWG is a symmetric shell, excited on second mode of its free vibrations. This vibrating shell is called resonator, and the excited free vibrations are actually static waves. As shown in Figure 1, the resonator is connected to the base of gyro through a solid bar. Excitation and establishment of the resonator vibrations are realized by an electrical actuator system. If gyroscope's case rotates, the generated static waves will precess with respect to the inertial space and the case. The amount of the precession of the static waves is proportional to the angular velocity of the gyro base. Spherical, cylindrical, and ring-thin shell resonators are normally used in this kind of gyro (Figure 2).

The main advantages of SWG are high accuracy, low weight/accuracy relation, low power, simple mechanical structure, stability relative to power disconnections, small start time, big dynamic range, low sensitivity against linear accelerations, and long life cycle. Based on the operational principles and applications in different gyroscopic systems, SWG can be divided into Solid-state Wave Rate Gyroscope (SWRG), and Solid-state Wave Integrating Gyro (SWIG). SWRG used to measure the angular velocity, has a low accuracy. On the contrary, SWIG can be manufactured to measure with high accuracy. For instance, a SWIG, which is being used for Cassini spacecraft control system, has an accuracy rate higher than 10^{-3} deg/hr.

SWG STRUCTURE AND OPERATION

It is assumed that the symmetrical thin shell resonator is excited in its second mode, and has a form of static wave. Peaks A, B, C, and D are shown in Figure 3-a. Since the resonator rotates with an

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angular velocity of Ω about its symmetrical axis, in points A, B, C and D we will have a complex motion *i.e.*, a relative motion with velocities $\bar{V}_A$,$\bar{V}_B$,$\bar{V}_C$,$\bar{V}_D$ respectively plus an additional angular motion with angular velocity of Ω. Coriolis acceleration of mass element in points A, B, C and D are quantified at $\bar{W}_{KA}$,$\bar{W}_{KB}$,$\bar{W}_{KC}$ and $\bar{W}_{KD}$ respectively. Coriolis inertial forces $\bar{P}_{KA}$,$\bar{P}_{KB}$,$\bar{P}_{KC}$,$\bar{P}_{KD}$ applied to points A, B, C and D, are in the opposite direction to the acceleration, and results in two couples. These two couples cause a rotation in the wave field (static waves) with respect to the resonator shell (Bryan's effect). By determining the redistribution of the amplitude of static waves relative to the resonator's body (gyro base), adequate information can be found about the angular rotation of SWG base in the inertial space. To recapitulate, the orientation of static waves in the second mode with respect to the resonator's body (angle ϑ) is a function of $\Omega(t)$ as well as some other resonator characteristics.

$$\vartheta(t) = f(\Omega,\, t\,, \ldots.) \tag{1}$$

Figure 4, shows new orientation of static wave after 90 of resonator's rotation. An actual example of SWG with hemispherical resonator is also shown in Figure 1.

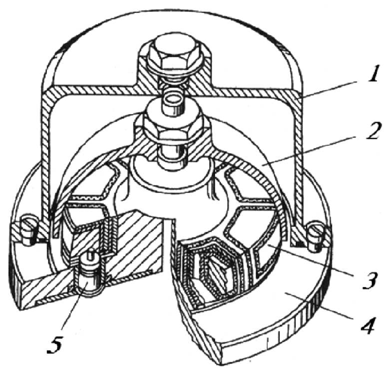

Figure 1. SWG with hemispherical resonator 1- vacuumed housing 2- resonator 3- pick off and excitation electrodes 4- base 5- vacuum exhaust

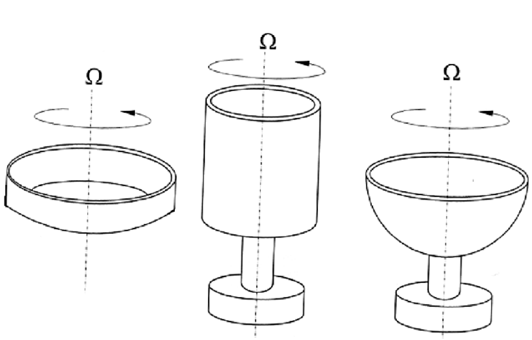

Figure 2. Thin shell resonators of SWG (hemispherical, cylindrical and ring)

EQUATION OF MOTION OF SWG WITH CYLINDRICAL RESONATOR

Cylindrical resonator for SWG, especially in the case of small size gyroscopes, has some advantages [4, 6, 10]. The hemispherical and ring resonators are studied as quoted in the reference list [4, 6, 7, 8, 9, 11 and 12] however, little research work is done on the cylindrical resonator. Therefore, this study represents a new approach to the analytical solution of dynamics of the cylindrical resonator.

The cylindrical model of the resonator is a thin cylindrical shell with upper edge oscillation in the horizontal plane. In the present study, the principle of equation of cylindrical shells is modified in view of the effects of very small dimensions of the cylindrical shell as well as the effects of rotation of the resonator.

The equations of motion of the resonator can be obtained based on the Kirshof-Liav theory [3] of thin shells. According to Figure 5, cylindrical coordinate system $R(r, \phi, z)$ with origin in the root of the resonator shell, can be considered as the reference coordinate system. The local coordinate system is called $R(u, v, w)$. Local coordinate u, v and w are in the longitudinal, circumferential and radial directions of the cylindrical mass element. Rotation of the gyro base is applied in the equations of motion as inertial force, applying to elements of the shell. The total acceleration of a mass element considering the angular velocity Ω is:

$$\vec{a} = \frac{d^2 r}{dt^2} = \vec{a}_0 + \vec{\ddot{r}} + \vec{\dot{\Omega}} \times \vec{r} + 2\vec{\Omega} \times \vec{\dot{r}} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{2}$$

The acceleration vector components are as follows:

$$a_u = \ddot{u}$$

$$a_v = \ddot{v} + \dot{\Omega}(R + w) - 2\Omega\dot{w} + \Omega^2 \nu$$

$$a_w = \ddot{w} + \dot{\Omega}\nu + 2\Omega\dot{\nu} + \Omega^2(R + w) \tag{3}$$

Based on D'Alambert Principle, taking into account the inertial forces obtained from acceleration components (3), the equations of motion of resonator's edge in altitude $Z = l$ will be:

$$\frac{\partial^2 u}{\partial z^2} + \left(\frac{1-\nu}{2R^2}\right).\frac{\partial^2 u}{\partial \phi^2} + \left(\frac{1+\nu}{2.R}\right).\frac{\partial}{\partial z}\frac{\partial v}{\partial \phi} + (\frac{\nu}{R}).\frac{\partial w}{\partial z} + (\frac{\nu^2-1}{E}).\rho.\frac{\partial^2 u}{\partial t^2} = 0 \tag{4}$$

$$\left(\frac{1+\nu}{2.R}\right).\frac{\partial}{\partial z}\frac{\partial u}{\partial \phi} + \left(\frac{1-\nu}{2}\right).\frac{\partial^2 v}{\partial z^2} + \left(\frac{1}{R^2}\right).\frac{\partial w}{\partial \phi} + \frac{1}{R^2}\frac{\partial^2 v}{\partial \phi^2} - \left(\frac{1-\nu^2}{E}\right).\rho.\left[\left(\frac{\partial^2 v}{\partial t^2}\right) - \left(2.\Omega.\frac{\partial w}{\partial t}\right)\right] = 0 \tag{5}$$

$$
\begin{aligned}
&(\frac{v}{R}).\frac{\partial u}{\partial z}+(\frac{1}{R^2}).\frac{\partial v}{\partial \phi}+\frac{w}{R^2}-\left(\frac{h^2}{12R^2}\right)\cdot\left[\left(R^2.\frac{\partial^4 w}{\partial z^4}\right)\right.\\
&\quad\left.+\left(2.\frac{\partial^2}{\partial z^2}\frac{\partial^2 w}{\partial \phi^2}\right)+\left(\left(\frac{1}{R^2}\right).\frac{\partial^4 w}{\partial \phi^4}\right)\right]\\
&\quad-\left(\frac{1-\nu^2}{E}\right).\rho.\left(\left(\frac{\partial^2 w}{\partial t^2}\right)+\left(2.\Omega.\frac{\partial v}{\partial t}\right)\right)=0 \qquad (6)
\end{aligned}
$$

Solution of the equations of motion using Boobnov-Galerkin method

The three first mode shapes of the resonator are shown in Figure 6. Modes number 0 and 1 aren't used in SWG, but the second mode, which has an elliptical shape, is applicable to our approach. For equations 4 to 6, the edge displacements for the second mode of the resonator can be derived [5] as follows:

$$
\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} U(z)\cos 2\phi \\ V(z)\sin 2\phi \\ W(z)\cos 2\phi \end{bmatrix} p(t) + \begin{bmatrix} U(z)\sin 2\phi \\ -V(z)\cos 2\phi \\ W(z)\sin 2\phi \end{bmatrix} q(t) \quad (7)
$$

In equation (7), $U(z) = V(z), W(z)$ are *Rayleigh* functions [3], determining vibrations of a non-flexible chamber in the second mode (Figure 5). Solving the

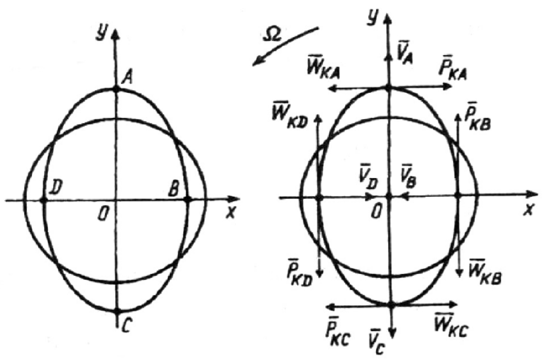

Figure 3. Mechanism of precession of the static waves in resonator a) non rotating resonator $\Omega = 0$. b) rotating resonator $\Omega = 0$.

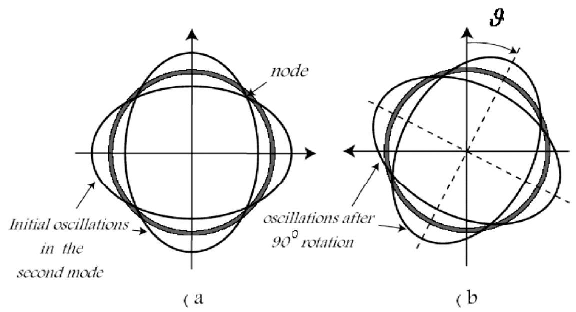

Figure 4. Precession of static waves with respect to the rotating resonator.

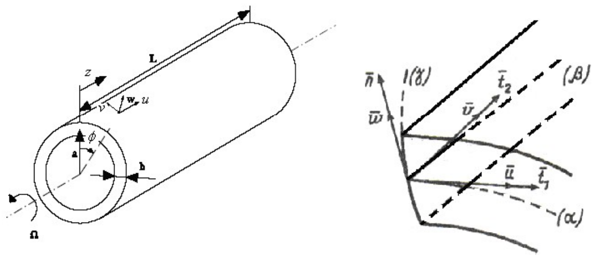

Figure 5. Cylindrical system of coordinates (r, ϕ, z) and positional system of coordinates (u, v, w) in longitudinal, circumferential and radial directions of the cylindrical mass element, $(\alpha), (\beta), (\gamma)$ are coordinate lines, $u, v, \vec{w}$ are components of displacement vector and $\vec{n}, \vec{t}_2, \vec{t}_1$ are unit vectors.

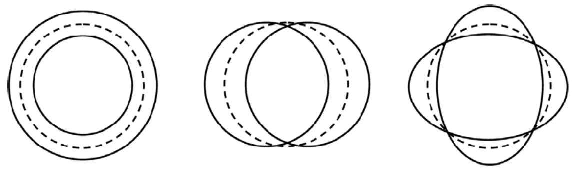

Figure 6. 3 first natural modes of resonator vibrations Dash lines are the original position of the resonator.

equations using the Boobnov-Galerkin method [5], the system of partial differential Eqs. (4), (5), (6) will be in the form of ordinary differential equations:

$$
\begin{cases} m_0\ddot{p}(t) - 2\Omega b\dot{q}(t) + c_0 p(t) = 0 \\ m_0\ddot{q}(t) + 2\Omega b\dot{p}(t) + c_0 q(t) = 0 \end{cases} \quad (8)
$$

where b, m_0 and c_0 are found based on the derived answer and *Rayleigh – Ritz* functions. Multiplying the first equation of (7) in i and adding it to the second equation, and using the complex variable $z(t) = p(t) + iq(t)$, we could obtain a standard form of:

$$
\ddot{z} + 2i\Omega\, a\, \dot{z} + \omega_0^2 z = 0 \quad (9)
$$

where $a = b/m_0$ and $\omega_0 = \sqrt{c_0/m_0}$ is the natural frequency of the resonator. The complete version of Eq. (9) can be written as:

$$
z(t) = e^{-i\alpha\Omega t}(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}) \quad (10)
$$

In Equation (10), C_1 and C_2 are complex coefficients, which are determined from initial conditions. In addition, Eq. (10) shows that radius vector of point z in the complex plane, rotates in the opposite direction of the resonator with a speed of $K.\Omega$. This means that the static wave gradually changes its orientation in the resonator's body with an angular velocity of $\dot{\vartheta} = -K\Omega$, where K is scalar factor (conversion factor) of the SWG, and can be obtained from the following relation:

$$K=\frac{b}{2m_0}=\frac{\int_0^l VWdz}{\int_0^l (U^2+V^2+W^2)dz}=0.30110330 \quad (11)$$

It could be found from Relation (11), that scale factor of gyro is constant for an ideal SWG, and it does not depend on the geometrical parameters, material or any other characteristics of gyroscope. For instance, for all cylindrical resonators, K always assumes the value of Relation (10). A series of test results of a SWG with cylindrical resonator is presented by Yatsenko *et.al.* [6]. The measured scale factor in this report is $K = 0.3$, which is very close to the analytical result presented in this paper. Yatsenko *et.al.* results show good confirmation of the analytical solution presented in this study.

Verification of the obtained results for K

Table 1, compares the scale factor for SWG, obtained from this study through other analytical and experimental research approaches undertaken in the field. The scale factor derived specifically from this study for cylindrical SWG approximately corresponds with the experimental results in [6]. Therefore, the related literature verifies the methods and supports the results of the present study.

MODELING OF EXCITATION OF RESONATOR VIBRATIONS IN SWG

Cylindrical resonator of SWG exited by external forces applied to edges

Equations of motion of cylindrical resonator, which is excited by external forces, according to equation 4, 5 and 6, will have the following forms [5]:

$$\frac{\partial^2 u}{\partial z^2}+\left(\frac{1-\nu}{2R^2}\right)\frac{\partial^2 u}{\partial \phi^2}+\left(\frac{1+\nu}{2R}\right)\frac{d^2}{\partial z\partial\phi}\nu$$
$$+(\frac{\nu}{R})\frac{\partial}{partialz}w+\left(\frac{\nu^2-1}{E}\right).\rho.\frac{\partial^2 u}{\partial t^2}+\xi\frac{\partial^2\dot{u}}{\partial z^2}$$
$$+\xi\frac{1+\nu}{2R}\frac{\partial^2\dot{\nu}}{\partial z\partial\phi}+\xi\frac{1-\nu}{2R^2}\frac{\partial^2\dot{u}}{\partial^2\phi}+\xi\frac{\nu}{R}\frac{\partial\dot{w}}{\partial z}=0 \quad (12)$$

Table 1. Founded scale factors of different SWG.

K	Type of resonator	References
0.311	sphere	S.A. Karapolof [1]
0.289	sphere	N.E. Yegarmin [1]
0.4	ring	V.F. Juravlief, D.M. Kelimof
0.312	sphere	V.F. Juravlief, D.M. Kelimof
0.295	sphere	experimental results for sphere [1]
0.30	cylinder	experimental results [6]
0.301	cylinder	obtained analytical result of this paper

$$\left(\frac{1+\nu}{2.R}\right).\frac{\partial}{\partial z}\frac{\partial}{\partial\phi}u+\left(\frac{1-\nu}{2}\right).\frac{\partial^2}{\partial z^2}v+\frac{1}{R^2}\frac{\partial^2 v}{\partial v^2}$$
$$+\left(\frac{1}{R^2}\right).\frac{\partial w}{\partial z}-\left(\frac{1-\nu^2}{E}\right).\rho.\left[\left(\frac{\partial^2 v}{\partial t^2}\right)-\left(2.\Omega.\frac{\partial w}{\partial t}\right)\right]$$
$$+\frac{\xi}{2R}(1+\nu)\frac{\partial^2\dot{u}}{\partial z\partial\phi}+\frac{1-\nu}{2}\xi\frac{\partial^2\dot{v}}{\partial z^2}+\frac{\xi}{R^2}\frac{\partial^2\dot{v}}{\partial\phi^2}+\frac{\xi}{R^2}\frac{\partial\dot{w}}{\partial\phi}=0 \quad (13)$$

$$(\frac{v}{R}).\frac{\partial u}{\partial z}+(\frac{1}{R^2}).\frac{\partial v}{\partial\phi}+\frac{w}{R^2}-\left(\frac{h^2}{12R^2}\right).$$
$$\left[\left(R^2.\frac{\partial^4 w}{\partial z^4}\right)+\left(2.\frac{\partial^2}{\partial z^2}\frac{\partial^2 w}{\partial\phi^2}\right)+\left(\left(\frac{1}{R^2}\right).\frac{\partial^4 w}{\partial\phi^4}\right)\right]$$
$$-(\frac{1-\nu^2}{E}).\rho.\left(\left(\frac{\partial^2 w}{\partial t^2}\right)+\left(2.\Omega.\frac{\partial v}{\partial t}\right)\right)+\left(\frac{\xi}{R^2}\right)\frac{\partial^2 v}{\partial t\partial\phi}$$
$$+\left(\frac{\xi}{R^2}\right)\frac{\partial w}{\partial t}+\left(\frac{\nu\xi}{R}\right)\frac{\partial^2 u}{\partial t\partial z}=\frac{p''_w-p'_v}{\rho\, s} \quad (14)$$

where R is the radius of original cylinder, $w(\phi,t)$ is vertical deformation of cylinder surface in time t, and p_v, p_w are radial and tangential components of distributed external force, with regard to original cylinder, and ξ is a coefficient that specifies the time of damping of free vibrations.

In Relations (12)-(14), the symbol "o" shows the time derivation and "′" shows derivation with respect to ϕ.

It was shown in [1] that Eqs. (12) - (14) describe the essential characteristics of static wave, excited in SWG resonator. In other words, it shows the precision of wave field as effected by the rotation of the cylinder, with an angular rate of Ω in its plane, when $\xi = 0$ and $p_w = p_v = 0$ (for ideal and free cylinder).

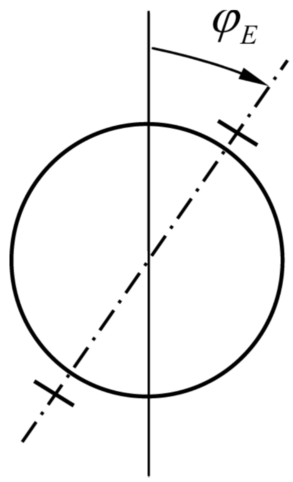

Figure 7. Angular position of electrodes in a positional excitation system .

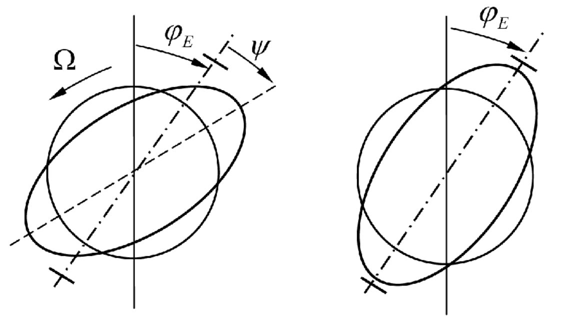

Figure 8. Static wave behavior a) non rotating resonator b) rotating resonator.

Positional excitation of resonator vibrations

In this section, a positional excitation of SWG resonator, based on cylindrical model of resonator is investigated and analyzed. The main form of the vibrations is considered in the second mode, were there are four peaks and four nodes.

The principle of operation and specifications of this system is described in Reference [5]. Mechanism of positional excitation is shown in Figure 7. A set of electrodes are mounted in the hosing and around the upper edge of the cylindrical shell in equal distance from the angular position. These electrodes generate a distributed external force applied to the resonator's edge. Each opposite couple of electrodes is excited by an AC electrical voltage, with a frequency of one half of the resonator[s second natural frequency. Front surface of the resonator and electrodes are covered by a thin layer of a conducting material, such as gold, which can make an electrical capacitor [1]. When the capacitor is charged, the two surfaces potentially try to pool each other, which causes a forced vibration. For a unit surface, this force can be determined by the following relation:

$$p = -\frac{\varepsilon_0}{2}\left(\frac{V}{d}\right)^2 \tag{15}$$

where V is the voltage between surfaces, d is the aerial distance between the surfaces, and ϵ_0 is the dielectric coefficient. Replacing voltages from Eq. (4), the vertical component of the external force will be:

$$p_w(\varphi, t) = -\frac{\varepsilon_0 L}{2d^2} V_0^2 f(\varphi)\cos^2\frac{\lambda}{2}t \tag{16}$$

Applying Eq. (16) to Eqs. (12)-(14), the following equations describing forced vibrations of the resonator[s edges can be derived:

$$\begin{aligned}&\frac{\partial^2 u}{\partial z^2} + \left(\frac{1-\nu}{2R^2}\right).\frac{\partial^2 u}{\partial \phi^2} + \left(\frac{1+\nu}{2.R}\right).\frac{\partial}{\partial z}\frac{\partial v}{\partial \phi} + \left(\frac{\nu}{R}\right).\frac{\partial w}{\partial z}\\ &\quad + \left(\frac{\nu^2-1}{E}\right).\rho.\frac{\partial^2 u}{\partial t^2} + \xi\frac{\partial^2 \dot{u}}{\partial z^2} + \xi\frac{1+\nu}{2R}\frac{\partial^2 \dot{v}}{\partial z \partial \phi}\\ &\quad + \xi\frac{1-\nu}{2R^2}\frac{\partial^2 \dot{u}}{\partial^2 \phi} + \xi\frac{\nu}{R}\frac{\partial \dot{w}}{\partial z} = 0\end{aligned} \tag{17}$$

$$\begin{aligned}&\left(\frac{1+\nu}{2.R}\right).\frac{\partial}{\partial z}\frac{\partial u}{\partial \phi} + \left(\frac{1-\nu}{2}\right).\frac{\partial^2 v}{\partial z^2} + \frac{1}{R^2}\frac{\partial^2 v}{\partial \phi^2}\\ &\quad + \left(\frac{1}{R^2}\right).\frac{\partial w}{\partial z} - \left(\frac{1-\nu^2}{E}\right).\rho.\left[\left(\frac{\partial^2 v}{\partial t^2}\right) - \left(2.\Omega.\partial\frac{dw}{\partial t}\right)\right]\\ &\quad + \frac{\xi}{2R}(1+\nu)\frac{\partial^2 \dot{u}}{\partial z \partial \phi} + \frac{1-\nu}{2}\xi\frac{\partial^2 \dot{v}}{\partial z^2} + \frac{\xi}{R^2}\frac{\partial^2 \dot{v}}{\partial \phi^2} + \frac{\xi}{R^2}\frac{\partial \dot{w}}{\partial \phi} = 0\end{aligned} \tag{18}$$

$$\begin{aligned}&\left(\frac{v}{R}\right).\frac{\partial u}{\partial z} + \left(\frac{1}{R^2}\right).\frac{\partial v}{\partial \phi} + \frac{w}{R^2} - \left(\frac{h^2}{12R^2}\right).\\ &\quad \left[\left(R^2.\frac{\partial^4 w}{\partial z^4}\right) + \left(2.\frac{\partial^2}{\partial z^2}\frac{\partial^2 w}{\partial \phi^2}\right) + \left(\left(\frac{1}{R^2}\right).\frac{\partial^4 w}{\partial \phi^4}\right)\right]\\ &\quad - \left(\frac{1-\nu^2}{E}\right).\rho.\left(\left(\frac{\partial^2 w}{\partial t^2}\right) + \left(2.\Omega.\frac{\partial v}{\partial t}\right)\right)\\ &\quad + \left(\frac{\xi}{R^2}\right)\frac{\partial^2 v}{\partial t \partial \phi} + \left(\frac{\xi}{R^2}\right)\frac{\partial w}{\partial t} + \left(\frac{\nu\xi}{R}\right)\frac{\partial^2 u}{\partial t \partial z}\\ &\quad = H\cos(2(\phi - \phi_E))\cos(\lambda t)\end{aligned} \tag{19}$$

where:

$$H = 2\varepsilon_0 V_0^2 L \sin\varphi_{El}/(\pi d_0^2 \rho S) \tag{20}$$

Steady state solution of Eqs. (17), (18) and (19) can be considered as fallows [5]:

$$\begin{aligned}u(z,\phi,t) &= U(z).\cos(2\phi).p(t) + U(z).\sin(2\phi).q(t)\\ v(z,\phi,t) &= V(z).\sin(2\phi).p(t) - V(z).\cos(2\phi).q(t)\\ w(z,\phi,t) &= W(z).\cos(2\phi).p(t) + W(z).\sin(2\phi).q(t)\end{aligned} \tag{21}$$

Using Eqs. (21), (17), (18) and (19) and employing Bubnov-Galerkin method [5], the following Equations would result:

$$\begin{cases}\ddot{p}(t) + \omega^2\xi\,\dot{p}(t) + \omega^2 p(t) - 4K\Omega\dot{q}(t) = \frac{H\cos 2\varphi_E \cos\lambda t}{J}\\ \ddot{q}(t) + \omega^2\xi\dot{q}(t) + \omega^2 q(t) + 4K\Omega\dot{p}(t) = \frac{H\sin 2\varphi_E\cos\lambda t}{J}\end{cases} \tag{22}$$

where $p(t)$ and $q(t)$ assume the following forms:

$$p(t) = a\cos\lambda t + m\sin\lambda t$$

$$q(t) = b\cos\lambda t + n\sin\lambda t \tag{23}$$

Consequently, the following equations may be used for determining a, b, m and n:

$$\begin{cases} m\omega_0^2\xi\lambda - 4K\Omega n\lambda = \frac{H\cos 2\varphi_E}{J} \\ n\omega_0^2\xi\lambda + 4K\Omega m\lambda = \frac{H\sin 2\varphi_E}{J} \\ a = b = 0 \end{cases} \tag{24}$$

m and n can be obtained by solving Eq. 24 and prove to be:

$$\begin{cases} n = -H\frac{(-\omega^2\xi\sin(2\phi_E)+4K\Omega\cos(2\phi_E))}{(\lambda J(\omega^4\xi^2+16K^2\Omega^2))} \\ m = H\frac{(\omega^2\xi\cos(2\phi_E)+4K\Omega\sin(2\phi_E))}{(\lambda J(\omega^4\xi^2+16K^2\Omega^2))} \end{cases} \tag{25}$$

also:

$$\frac{n}{m} = \frac{\omega^2\xi\sin(2\phi_E) - 4K\Omega\cos(2\phi_E)}{\omega^2\xi\cos(2\phi_E) + 4K\Omega\sin(2\phi_E)}$$

$$= \frac{\omega^2\xi\tan(2\phi_E) - 4K\Omega}{\omega^2\xi + 4K\Omega\tan(2\phi_E)} \tag{26}$$

In addition, $w(\phi, t)$ for resonator[s edge deformations will have the following relation:

$$w(\varphi, t) = W(l)\sin(\omega_0 t)(m\cos 2\varphi + n\sin 2\varphi) \tag{27}$$

Eq. (27) can be simplified to:

$$w(\varphi, t) = W(l)\sqrt{m^2+n^2}\sin\omega_0 t\cos 2(\varphi - \vartheta) \tag{28}$$

where:

$$\tan 2\vartheta = n/m \tag{29}$$

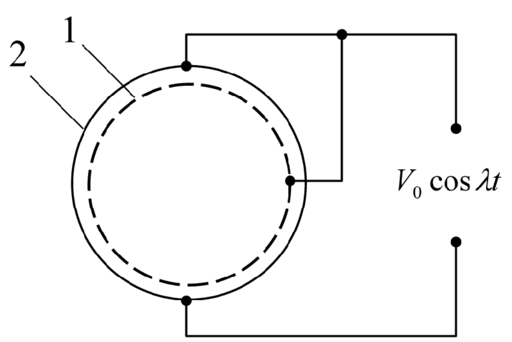

Figure 9. The ring electrode in a parametric excitation system 1-resonator, 2-the ring electrode.

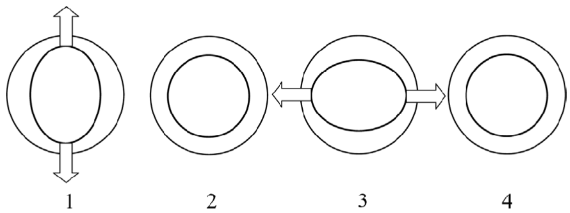

Figure 10. Parametric excitation process.

Calculation of phase

Determining m and n from Eq. (24) and replacing them into Relation (29) angle ϑ (orientation of static wave) for positional excitation would be:

$$\tan 2\vartheta = \frac{\omega^2\xi\tan(2\phi_E) + 4K\Omega}{\omega^2\xi + 4K\Omega\tan(2\phi_E)} \tag{30}$$

By using just the first order of the Tailor Series for Eq. (30), and by ignoring small values of Ω^2, we will have:

$$\tan 2\vartheta = \tan 2\varphi_E - \frac{4K\Omega}{\omega^2\xi}(1 + \tan^2 2\varphi_E) \tag{31}$$

Using the first order of Tailor Series for $\tan(2\phi_E)$ and $\tan(2\vartheta)$ in terms of ϕ_E and ϑ, we will have:

$$2\vartheta = 2\phi_E - \frac{4K\Omega}{\omega^2\xi} \Rightarrow 2\vartheta = 2\phi_E - 2\psi \Rightarrow \psi = \frac{2K\Omega}{\omega^2\xi} \tag{32}$$

Analysis of Relation (32) shows that: if $\Omega = 0$, then orientation of wave field in the SWG resonator is constant, *i.e.* static wave is attached to the resonator's body (Figure 8-a). If $\Omega = 0$, and the peak of the static wave will go back from its original orientation. The amount of this displacement is $\vartheta = \varphi_E - \psi$ (Figure 8-b), where:

$$\psi = 2.K.\frac{\Omega}{\omega_0^2\xi}$$

Parametric excitation of resonator vibrations

In this section, the principle of the parametric excitation system of SWG resonator is investigated for when the gyroscope has an integrating roll in body rotation sensing.

Parametric excitation is generated by a ring electrode around the resonator's edge. Surfaces of the resonator and the ring electrode make a cylindrical capacitor, which is charged by a voltage, that does not depend on ϕ, and has a frequency close to the second natural frequency of the resonator:

$$V = V_0\cos\lambda t \tag{33}$$

The role of the parametric excitation system is to compensate for the energy dissipation of the resonator, which occurs because of the structural damping and frictions in the resonator and in addition, because of the residual gas in the vacuumed space of gyroscope.

The mechanism for parametric excitation is shown in Figure 10. When there is no deformation in the resonator shell, the electrical forces of capacitor cause the internal stresses. When the resonator starts to deform where aerial space is decreasing, the local forces increase. In the section with 90° phase aerial space increase, however, local forces decrease. (In fact, the force is proportional to the inverse of square of the distance between the electrode and the resonator). Resultant force acts in direction of peaks of static wave (main axis of the ellipse) and causes more deformation of the resonator.

Figure 10 shows the above process. Part 1 of Figure 10 shows that the resonator moves to its maximum deformation. Power voltage V_0 is also in its maximum. Part 2 of this figure shows that static waves are restored to their equilibrium status due to inertia. In this case, power voltage is off. Parts 3 and 4 of this figure demonstrate that this cycle is repeated, but in the opposite direction.

Solution of equations of motion of gyroscope with disturbing external forces

It can be assumed that the tangential component of the electrical force is negligible, and there is just radial force. To apply this force to the equations of motion, we can write the first order polynomial Series of p_w as follows:

$$p_w = -\frac{\varepsilon_0}{2}\frac{V^2}{(d_0+w)^2} \approx \frac{\varepsilon_0 V^2 w}{d_0^3} + ... \tag{34}$$

Substituting Eq. (34) into Eq. (4), the final equations of motion are derived as follows:

$$\begin{aligned}&(\frac{v}{R}).\frac{\partial u}{\partial z} + (\frac{1}{R^2}).\frac{\partial v}{\partial \phi} + \frac{w}{R^2} - \left(\frac{h^2}{12R^2}\right). \\ &\quad \left[\left(R^2.\frac{\partial^4 w}{\partial z^4}\right) + \left(2.\frac{\partial^2}{\partial z^2}\frac{\partial^2 w}{\partial \phi^2}\right) + \left(\left(\frac{1}{R^2}\right).\frac{\partial^4 w}{\partial \phi^4}\right)\right] \\ &\quad -(\frac{1-\nu^2}{E}).\rho.\left(\left(\frac{\partial^2 w}{\partial t^2}\right) + \left(2.\Omega.\frac{\partial v}{\partial t}\right)\right) \\ &\quad + \left(\frac{\xi}{R^2}\right)\frac{\partial^2 v}{\partial t \partial \phi} + \left(\frac{\xi}{R^2}\right)\frac{\partial w}{\partial t} + \left(\frac{\nu\xi}{R}\right)\partial\frac{\partial^2 u}{dt\partial z} \\ &\quad = w'' R\cos^2(\lambda\, t)\end{aligned} \tag{35}$$

where:

$$R = \frac{\varepsilon_0 L V_0^2}{\rho.S.d_0^2}$$

Using Eq. (32) and (33) and applying the Bubnov-Galerkin method, the following system of equations can be obtained:

$$\begin{aligned}&\ddot{p}(t)+\omega^2\xi\dot{p}(t)+\omega^2 p(t)-4K\Omega\dot{q}(t) = \frac{4Rl_E p(t)\pi\cos^2(\lambda t)}{J} \\ &\ddot{q}(t)+\omega^2\xi\dot{q}(t)+\omega^2 q(t)+4K\Omega\dot{p}(t) = \frac{4Rl_E q(t)\pi\cos^2(\lambda t)}{J}\end{aligned} \tag{36}$$

By averaging Eq. (36) in one period of the oscillations, we can simplify the vibrating process in the resonator as:

$$\begin{aligned}w(\varphi,t) &= W(l)(a\cos\lambda\, t + m\sin\lambda\, t)\cos 2\varphi \\ &\quad + (b\cos\lambda\, t + n\sin\lambda\, t)\sin 2\varphi\end{aligned} \tag{37}$$

Eq. (37) describes a static wave, given the following condition:

$$\det = \begin{bmatrix} a & m \\ b & n \end{bmatrix} = 0 \tag{38}$$

Using functions a, b, c, d in [5], relation (37) can be rewritten as:

$$w(\varphi,t) = \sqrt{a^2+m^2+b^2+n^2}\cos(\lambda\, t-\alpha)\cos 2(\varphi-\vartheta) \tag{39}$$

where $\tan(2\vartheta) = \sqrt{b^2+n^2}/\sqrt{a^2+m^2}$, $\tan(\alpha) = m/a$ and ϑ determine orientation of static wave of Eq. (39), with respect to the resonator. For determining the variation of this angle, we can write the following relation:

$$\frac{d\vartheta}{dt} = \frac{1}{2}\frac{d}{dt}\left[\arctan\frac{\sqrt{b^2+n^2}}{\sqrt{a^2+m^2}}\right] \tag{40}$$

which can be simplified as [5]:

$$\frac{d\vartheta}{dt} \cong -K\Omega \tag{41}$$

$$\vartheta = \vartheta_0 - K\int_0^t \Omega(\tau)d\tau \tag{42}$$

where $k = 0.301$ is the scale factor of the cylindrical resonator, which is obtained in section 2 of this paper. The obtained result shows that parametric excitation makes no disturbance in the static waves of the resonator, and the gyroscope is completely free. In other words, the precision of the static wave is proportional to the angle of rotation of the gyro case. The SWG, therefore, works in an integrating regime.

CONCLUSION

In this research, a series of equations of motion of a Solid State Wave Gyroscope with rotary thin cylindrical shell resonator are developed. As a result of solving the equations of motion, the obtained scale factor for the cylindrical SWG is 0.301, which corresponds to the experimental results in the literature.

Two kinds of resonator excitations, positional and parametric, are investigated. Excitation forces, which are produced with electrodes, are applied in equations of motion as external forces. After solving the equations of motion, it is shown that the SWG with positional excitation is a Rate gyroscope, and the SWG with parametric excitation would be a Rate integrating gyro.

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