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Airline Stochastic Capacity Allocation by Applying Revenue Management

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To formulate a single-leg seat inventory control problem in an airline ticket sales system, the concept and techniques of revenue management are applied in this research. In this model, it is assumed that the cabin capacity is stochastic and hence its exact size cannot be forecasted in advance, at the time of planning. There are two groups of early-reserving and late-purchasing customers demanding this capacity. The price rate as well as the penalty for booking cancellation caused by overbooking is different for each group. The model is developed mathematically and we propose an analytical solution method. The properties of the optimal solution as well as the behavior of objective function are also analyzed. The objective function is neither concave nor convex in general. However, we prove it is a unimodal function and by taking advantage of this property, the optimal solution is determined.

INTRODUCTION

Revenue management is defined as the set of techniques used to assign the proper amount of a perishable asset to the right customers at the right time. Any revenue management problem contains the following common characteristics:

- A limited capacity of a perishable asset (like airline tickets) which cannot be enhanced easily in short term.
- Stochastic demand for perishable asset.
- Different customer groups. The available perishable asset can be sold at different prices, through different booking groups (usually at different periods).

In the models discussed in the literature, it is a common assumption that the amount of available capacity is known and deterministic, although perishable. This assumption can be justified for most of the cases in airlines, hotels, service industries, but not necessarily for all situations. For example, in many real world airline ticket sales systems, the capacity is stochastic in nature due to some unanticipated/forced group reservations or some technical reasons. Such cases are known as shifting capacity or moving curtain in the literature.

In many ticket sales systems, acceptance or rejection of a booking request depends on the availability of capacity as well as on its price. If a reservation is made but then cancelled, the system has to pay some penalty. This may happen if the required capacity for the booked tickets exceeds the available capacity, due to its stochastic nature. On the other hand, rejecting a booking request means loss of income. Therefore, the acceptance or rejection of a request results in a tradeoff between income loss and penalty cost. Then, the aim is to develop an optimal policy of acceptation/rejection of booking requests in order to earn the maximum expected total profit (including both the sales income and the penalty cost). The complexity arises from the fact that the capacity is not known at the time of reservation, due to its stochastic nature.

We assume there are two groups of customers, *i.e.*, late-purchasing and early-reserving customers. The first group has priority and their expense is usually (but not necessarily) less than that of the customers of the second group. Since the price rate and cancellation penalty cost for each group is different, the acceptance/rejection policy for each group also differs appropriately. In fact, the optimal policy determines the maximum capacity that can be assigned to each customer group. To formulate the model and obtain an optimal solution, we apply the concept of revenue

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management (RM) and modify some of its techniques accordingly to fit our problem.

This paper is organized as follows. The literature is reviewed briefly in the next section. We define the problem in more detail in "Capacity allocation problem" section. A model is developed in section 4 for capacity allocation problem with two customer groups and stochastic production capacity. Furthermore, in section 4, the unimodality property of the objective function is proved and an approach for determining the solution is also introduced. In the next section, entitled "Special case: Deterministic capacity", compatibility of the results against previous works is investigated. Sensitivity of the optimal solution and optimal value of the objective function with respect to various parameters is studied in "Sensitivity analysis" section. Finally, in the last section, "Conclusion", the paper is summarized and suggestions for future works are presented.

LITERATURE REVIEW

Revenue Management (also called Yield Management), which originated from airline industry refers to a set of methods, techniques, as well as concepts used to assign some perishable asset to several groups of customers. Littlewood [1] from former BOAC airline company, is usually considered as the founder of revenue management. He presented an analytical model to assign seats to two price groups of passengers in a flight. McGill and VanRyzin [2] categorized the major areas of research in the field of Revenue Management as seat inventory control, overbooking, pricing and demand forecasting. A review of mathematical optimization and operational research techniques in airline seat inventory control (both static and dynamic) is carried out by Pak and Piersma [3]. There is a vast literature regarding revenue management, although to our knowledge there is no model similar to ours.

The most familiar and oldest application of revenue management is in airline industry, where a fixed capacity of seats must be sold (booked) before each flight departure. However, it also has been effectively applied to other areas such as car rental, Geraghty and Johnson [4], Carol and Grimes [5], broadcasting (Cross [6]), cruise ships (Kimes [7], Belobaba [8], Ladany and Arbel [9], Smith *et.al.* [10], Gallego and van Ryzin [11]), Internet service providers (Nair *et.al.* [12], Paschalidis and Tsitiklis [13]), railways (Kimes [7], Strasser [14], Ciancimino et al. [15], Wen *et.al.* [16], Chen and Gao [17]), nonprofit sector (Metters and Vargas [18], Kraft [19]), lodging and hotels (Rothstein [20], Ladany [21], Liberman and Yechiali [22,23], Kimes [7], Hanks *et.al.* [24], Bitran and Mondschein [25], Feng and Gallego [26], Bitran and Gilbert [27], Baker and Collier [28], Badinelli [29], Upchurch *et.al.* [30], Choi and Mattila [31]), restaurants (Kimes *et.al.* [32], Kimes [33], Kimes et al. [34], Kimes [35], Kimes and Thompson [36], Lai and Ng [37]), health care (Chapman and Carmel [38]), tourism (Shwartz and Lin [39]), holiday retail shopping (Coulter [40]) and production planning (Harris and Pinder [41], Modarres and Sharifyazdi [42], Modarres and Nazemi [43]).

Revenue management practitioners agree that RM is more applicable for environments with short-run fixed capacity (Smith *et.al.* [10], Cross [6], Harris and Pinder [41]). However due to many reasons, as mentioned before, the capacity may have a stochastic nature. Thus, in this paper we propose a two-group revenue management (seat inventory control) model with probabilistic capacity.

Despite the differences mentioned above, models dealing with the concept of expected marginal seat revenue, such as littlewood [1] and belobaba [8] can be considered as the ancestors of our model.

CAPACITY ALLOCATION PROBLEM

In this section, we define the problem in more detail. In subsequent sections, this problem is formulated mathematically and then we present our approach to obtain its optimal solution.

PROBLEM DEFINITION

Consider an airline ticket sales system with stochastic capacity. As mentioned before, the system has two types of customers, for example, late-purchasing and early-reserving customers. A booking limit is the maximum number of tickets, which can be assigned to a specific group. In the case of limited and deterministic capacity, there is a booking limit for the second group of customers while there is no limit for the first group as far as the acceptance of booking requests is concerned. However, the second group customers take advantage of some kind of discount and their price rate is usually less than that of the customers of the first group.

Demand (or total size of booking requests) in each group is a non-negative random variable with a known continuous probability distribution function and independent of demand of the other group.

Each booking request is either accepted or rejected at the arrival time, according to the adopted policy of the system. However, if an order is accepted and then cancelled by the airline at departure time, the system has to pay some penalty. Therefore, it is important to determine how many requests can be accepted from each group in order to maximize the expected total income (including the penalty cost).The price rate depends on the group of customer. Similarly, the cancellation penalty of a reservation differs, depending on the customer group. It is assumed that the penalty rate of booking cancellation for the second group customers is higher than that of the first group, because of their long term relationships.

The decision maker problem is how to allocate the available cabin capacity to the different booking groups. In other words, the optimal policy must determine the booking limit for the second group customers, while the capacity is not known at the time of reservation.

In the previous two-group revenue management models in the literature, a protection level is set for more desirable groups against a booking limit for the other groups. A protection level is the fraction of capacity that is kept only for the customers of a special group. In this model, we cannot determine any protection level for a group because the total capacity is not known at the time of planning.

PRIORITY RULE

At departure time, if the available capacity is less than the number of sold tickets, then the early-reserving customers (second group) have priority over the other group. In other words, when the capacity is not sufficient for all of the ticket-holding show-up customers, then the tickets of the first group customers are cancelled first. Tickets of the second group customers are canceled only if all sold tickets of the first group have been cancelled.

NOTATION OF INPUT DATA IN THE MODEL

r_1, r_2: price of a ticket for group 1 and 2, respectively;
p_1, p_2 : penalty of booking cancellation for group 1 and 2, respectively;
π_1, π_2 : lost profit of booking cancellation in group 1 and 2, respectively.
Note:

1. Logically $r_2 < r_1$ and we assume $p_1 < p_2$. However, the objective function and the method are not directly influenced by this assumption.
2. π_1, π_2 are not independent parameters and are derived from r_1, r_2, p_1, p_2, as follows:

$$\pi_i = r_i + p_i\,, \qquad i = 1, 2 \tag{1}$$

It is assumed that $\pi_1 < \pi_2$.

RANDOM VARIABLES:

x_1, x_2 : demand for tickets of group 1 and 2, respectively, $(x_1, x_2 \geq 0)$;
c : stochastic capacity of cabin. $(c \geq 0)$;
$f_1(x_1), f_2(x_2), f_c(c)$: probability density functions of x_1, x_2 and c, respectively;
$F_1(x_1), F_2(x_2), F_c(c)$: cumulative distribution functions for x_1, x_2 and c, respectively;
$\bar{F}_1(.) = 1 - F(.)$.

DECISION VARIABLES:

b_2: booking limits allocated to reservations of group 2;
a_1, a_2: the number of accepted reservation requests for groups 1 and 2, respectively;
d_1, d_2 : the number of denied reservations due to shortage, for customers of groups 1 and 2, respectively;
R : total revenue gained from income of ticket sales minus penalty of booking cancellation.

$$\begin{cases} a_1 = x_1 \\ a_2 = \min\{x_2,\, b_2\} \end{cases} \tag{2}$$

The number of cancelled tickets of group 2 customers depends on the capacity only. It is independent of the reservations of the first group, as explained by the "priority rule".

$$d_2 = \max\{0,\, a_2 - c\} \tag{3}$$

However, the number of cancelled tickets of group 1 customers depends on the capacity as well as the number of reservations for the second group, as mentioned in the "priority rule".

$$d_1 = \begin{cases} 0 & \text{if } \ a_1 + a_2 \leq c \\ a_1 + a_2 - c & \text{if } \ a_2 \leq c < a_1 + a_2 \\ a_1 & \text{if } \ 0 \leq c < a_2 \end{cases} \tag{4}$$

On the other hand, the objective function is $R = r_1 a_1 + r_2 a_2 - \pi_1 d_1 - \pi_2 d_2$. Therefore,

$$E(R) = r_1 E(a_1) + r_2 E(a_2) - \pi_1 E(d_1) - \pi_2 E(d_2) \tag{5}$$

Although the objective function in our model is to maximize the expected total revenue (minus penalty), it is also possible to consider other objectives such as maximizing capacity utilization, maximizing average revenue per customer, minimizing lost customer good will, minimizing opportunity cost (see McGill and Van Ryzin [2] or Pak and Piersma [3]).

The number of accepted booking requests, $E(a_i)$, $i = 1$, 2, is obtained from (2), as follows:

$$\begin{cases} E(a_1) = \int_0^\infty x_1 f_1(x_1)\, dx_1 \\ \\ E(a_2) = \int_0^{b_2} x_2 f_2(x_2)\, dx_2 + \int_{b_2}^\infty b_2 f_2(x_2)\, dx_2 \end{cases} \tag{6}$$

The expected size of capacity for denied reservations, $E(d_i)$, $i = 1$, 2, is obtained from (3) and (4), as follows:

$$E(d_1) = \int_{b_2}^{b_2+x_1}\int_{b_2}^{\infty}\int_0^{\infty} A + \int_{x_2}^{x_2+x_1}\int_0^{b_2}\int_0^{\infty} B$$

$$+ \int_0^{b_2}\int_{b_2}^{\infty}\int_0^{\infty} x_1 f_1(x_1) f_2(x_2) f_c(c) dx_1\, dx_2\, dc$$

$$+ \int_0^{x_2}\int_0^{b_2}\int_0^{\infty} x_1 f_1(x_1) f_2(x_2) f_c(c) dx_1\, dx_2\, dc$$

$$A = (x_1 + b_2 - c) f_1(x_1) f_2(x_2) f_c(c) dx_1\, dx_2\, dc$$

$$B = (x_1 + x_2 - c) f_1(x_1) f_2(x_2) f_c(c) dx_1\, dx_2\, dc \tag{7}$$

and similarly,

$$E(d_2) = \int_0^{x_2}\int_0^{b_2} (x_2 - c) f_2(x_2) f_c(c)\, dx_2\, dc$$

$$+ \int_0^{b_2}\int_{b_2}^{\infty} (b_2 - c) f_2(x_2) f_c(c)\, dx_2\, dc \tag{8}$$

OPTIMALITY CONDITION

In general, the objective function $E(R)$ is neither convex nor concave. This fact is verified through a counter example in appendix B. However, it will be proved that $E(R)$ is a unimodal function. Thus, this function has a maximum, which can lie on either boundary of the feasible region or zero point of first degree derivative of the objective function. As a result, the optimal solution is obtained by setting the derivative of $E(R)$ with respect to b_2 equal to zero, provided this set of equations has a solution. Otherwise, the optimal solution is zero.

UNIMODALITY PROPERTY OF OBJECTIVE FUNCTION

To prove $E(R)$ is a unimodal function, we present the following lemma. But first, it is assumed that some boundary properties hold for probability density functions as follows:

$$\begin{cases} \lim_{x_i\to\infty} f_i(x_i) = 0, \quad i = 1, 2 \\ \lim_{c\to\infty} f_c(c) = 0 \end{cases} \tag{9}$$

These assumptions are very common and rational and not restrictive in real world problems, because neither demand nor capacity can be unlimited.

Lemma 1. Derivative of the objective function with respect to b_2, *i.e.* ($\frac{\partial E(R)}{\partial b_2}$), is a product of a non-increasing function of b_2 and a non-negative non-increasing function of b_2.

Proof: As calculated in appendix A, $\frac{\partial E(R)}{\partial b_2} = \bar{F}_2(b_2).\psi(b_2)$, where:

$$\psi(b_2) = r_2 - (\pi_2 - \pi_1) F_c(b_2) - \pi_1 E_{x_1}[F_c(x_1 + b_2)] \tag{10}$$

while $E_{x_1}[F_c(x_1 + b_2)]$ is the expected value of $\Pr\{c \leq x_1 + b_2\}$ over x_1.

In our assumption $\pi_2 \geq \pi_1 \geq 0$ and also $F_c(b_2)$ and $E_{x_1}[F_c(x_1+b_2)]$ are non-decreasing functions of b_2. Hence, $\psi(b_2)$ is a non-increasing function of b_2. Thus, $\frac{\partial E(R)}{\partial b_2}$ is a product of two non-increasing functions of b_2, one of which ($\bar{F}_2(b_2)$) is non-negative.

Now, unimodality property of $E(R)$ will be proved during three succeeding theorems studying 3 mutually exclusive and comprehensive cases.

Theorem 1. Let $\psi(b_2) \geq 0$. Then, $E(R)$ is a unimodal function with respect to b_2, and attains its maximum at infinity. Figure 1 shows the typical curve of this function with respect to b_2.

Proof: Since both $\psi(b_2)$ and $\bar{F}_2(b_2)$ are non-negative, then the derivative is non-negative and $E(R)$ is non-decreasing. According to Appendix C, $\frac{\partial E(R)}{\partial b_2}$ approaches 0 at infinity. So, $E(R)$ increases till it reaches (tends to) a fixed value (peak) when b_2 approaches infinity. Clearly, in this case the optimal value of b_2 is found by setting $\frac{\partial E(R)}{\partial (b_2)}$ equal to zero and approaching infinity. That is there should be no limit for reservations in the second group. The structure of $\frac{\partial E(R)}{\partial (b_2)}$ in this case is illustrated in Figure 2.

Theorem 2. Let $\psi(b_2) > 0$ at $b_2 = 0$ and $\psi(b_2) < 0$ for big values of b_2 (in other words, the sign of $\psi(b_2)$ changes as b_2 raises). Then, $E(R)$ is a unimodal function with respect to b_2 and attains its maximum at the point where $\frac{\partial E(R)}{\partial (b_2)} = 0$. Figure 3 shows the typical curve of this function with respect to b_2.

Proof: Since at 0 the partial derivative is pos-

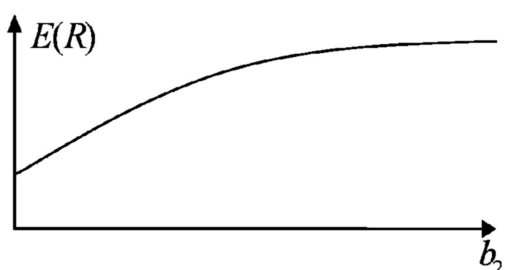

Figure 1. First case for E(R) curve.

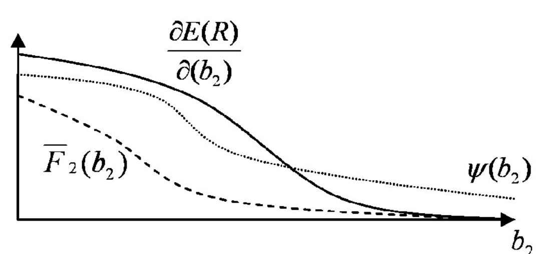

Figure 2. First case for partial derivative of E(R) with respect to b_2.

itive, then $E(R)$ has an ascending start. It increases till it reaches a peak when $\psi(b_2)$ equals zero and consequently $\frac{\partial E(R)}{\partial(b_2)} = 0$. Then, it starts to fall down because $\bar{F}_2(b_2)$ is positive and $\psi(b_2)$ is negative. Finally, it approaches a fixed value because $\bar{F}_2(b_2)$ approaches zero and as a result, the derivative approaches zero too. As mentioned before, this issue has been studied in appendix C. Clearly, in this case the optimal value of b_2 is found by setting $\frac{\partial E(R)}{\partial(b_2)}$ to zero. Figure 4 shows the typical structure of $\frac{\partial E(R)}{\partial(b_2)}$ in this case.

Theorem 3. Let $\psi_1(b_1) \leq 0$, then $E(R)$ is a unimodal function with respect to b_2 and attains its maximum at $b_2 = 0$. Figure 5 shows a typical curve of this function with respect to b_2.

Proof: Since at 0, the partial derivative is non-positive then $E(R)$ has a descending start. It approaches a fixed value for large values of b_2, because $\bar{F}_2(b_2)$ approaches zero (see Appendix C) and makes the derivative approach zero, too. Clearly, in this case, $b_2 = 0$ is the optimal value. For better illustration, nature of $\frac{\partial E(R)}{\partial(b_2)}$ is shown in Figure 6.

Note: Since $\lim_{b_2\to\infty} \psi(b_2) = r_2 - \pi_2 = -p_2 < 0$, the first case cannot take place. However, in the second case, the optimal value (zero point of derivative) may approach infinity.

OPTIMAL SOLUTION

Let b_2^* be the optimal value of booking limit for group 2. Concluding from the above cases, and also by considering the fact that $E(R)$ is a unimodal function, b_2^* is obtained through the following steps.

Step 1. Let β_2 be the solution of the equation $\frac{\partial E(R)}{\partial(b_2)} = 0$.

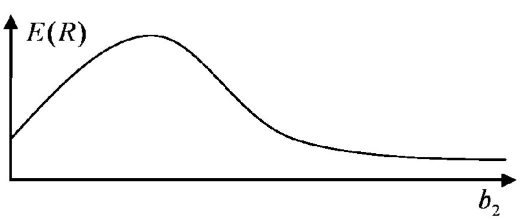

Figure 3. Second case for E(R) curve.

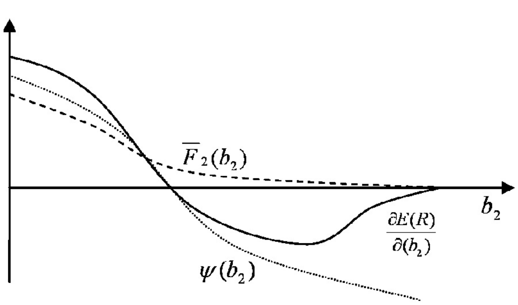

Figure 4. Second case for partial derivative of E(R) with respect to b_2.

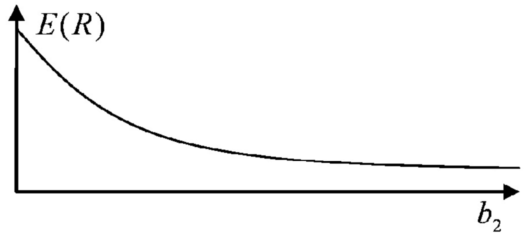

Figure 5. Third case for $E(R)$ curve.

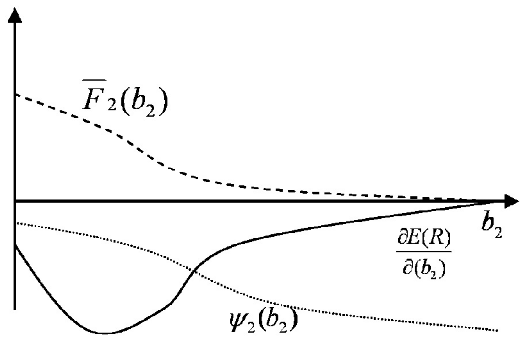

Figure 6. Third case for partial derivative of E(R) with respect to b_2.

Step 2. If $r_2/\pi_1 \geq E_{x_1}[F_c(x_1)]$ (or in other words $\beta_2 \geq 0$), then $b_2^* = \beta_2$.

Step 3. If $r_2/\pi_1 < E_{x_1}[F_c(x_1)]$ (or in other words $\beta_2 < 0$), then $b_2^* = 0$.

Proof: $\psi(0) = r_2 - \pi_1.E_{x_1}[F_c(x_1)]$. Therefore, if $r_2/\pi_1 \geq E_{x_1}[F_c(x_1)]$, then $\psi(0) \geq 0$ and we should refer to case 2. Otherwise, $\psi(0) < 0$ and the optimal solution can be found considering case 3.

SPECIAL CASE: DETERMINISTIC CAPACITY

To check the result of our model, consider a special case in which the capacity is deterministic and equal to $\hat{c} > 0$. In other words, c has a one-point probability distribution. Then, $\psi(b_2)$ can be rewritten as follows:

$$\psi(b_2) = \begin{cases} r_2 - \pi_1 \bar{F}_1\left(\hat{c} - b_2\right), & \text{if } b_2 < \hat{c} \\ r_2 - \pi_2 = -p_2 < 0, & \text{if } b_2 \geq \hat{c} \end{cases} \tag{11}$$

Since $\psi(b_2)$ is a non-increasing function, then a root of $\psi(b_2) = 0$ lies within $[0, \hat{c}]$. Thus, the optimal solution (b_2^*) is obtained by solving $\psi(b_2) = 0$, as follows:

$$\bar{F}_1(\hat{c} - b_2^*) = \frac{r_2}{\pi_1} \tag{12}$$

In our assumptions, $r_2 < \pi_2 \leq \pi_1$, then $0 \leq r_2/\pi_1 \leq 1$. Hence, the optimal solution can be found appropriately.

Littlewood [1] shows that in a two-group seat inventory control problem, when the capacity is fixed and deterministic (non-stochastic) and customer groups are nested ($i.e.$ no booking request belonging to the higher priced group is rejected to keep a seat for customers of a lower priced group), low-fare booking requests should be accepted as long as $r_2 \geq r_1.\Pr\{x_1 > P_1\}$, while P_1 is the protection level of group 1. Belobaba [8] extends this Littlewood's rule to multiple nested fare classes and introduces the term "Expected Marginal Seat Revenue" (EMSR) for the general approach. If the capacity is fixed, then $\hat{c} - b_2 = P_1$. In addition, if there is no penalty for cancellation of a booking due to shortage of capacity, then $\pi_1 = r_1 + p_1 = r_1$. So, relation (12) can be rewritten as:

$$\bar{F}_1(P_1^*) = \Pr\{x_1 > P_1^*\} = \frac{r_2}{r_1} \tag{13}$$

which is fully compatible with the results of EMSRa model (Belobaba [8])) for two groups as well as Littlewood [1] and Modarres and Sharifyazdi [42].

SENSITIVITY ANALYSIS

In this section, the sensitivity of the objective function as well as the optimal solution is studied with respect to expense parameters (r_1, r_2, p_1 and p_2) as well as parameters of probability distributions of demand (x_1 and x_2) and capacity (c). From:

$$E(R) = \sum_{i=1}^{2} [r_i(E(a_i) - E(d_i)) - p_i E(d_i)] \tag{14}$$

$E(R)$ is an increasing function of r_1 as well as r_2 and also a decreasing function of p_1 as well as p_2, by considering $E(a_i) \geq E(d_i), \forall i$.

However, we show that the optimal solution is sensitive to r_2/π_1, rather than to single parameters. From (10):

$$\psi(0) = r_2 - \pi_1 E_{x_1}[F_c(x_1)] \tag{15}$$

If r_2/π_1 is less than $E_{x_1}[F_c(x_1)]$, then the best value for b_2 is zero by Theorem 3. Conversely, if $r_2/\pi_1 \geq E_{x_1}[F_c(x_1)]$, then the optimal value of b_2 is the positive root of $\psi(b_2) = 0$. Note that $E_{x_1}[F_c(x_1)]$ depends on the parameters of probability distributions of capacity and the demand of first group customers, but not on other parameters. Also from (10):

$$\begin{aligned}\psi(b_2) &= r_2\bar{F}_c(b_2) - p_2 F_c(b_2) \\ &\quad - \pi_1[-F_c(b_2) + E_{x_1}[F_c(x_1 + b_2)]]\end{aligned} \tag{16}$$

Since $\bar{F}_c(b_2)$, $F_c(b_2)$ and $[-F_c(b_2) + E_{x_1}[F_c(x_1 + b_2)]]$ are non-negative and $\psi(b_2)$ is a non-increasing function, when $r_2/\pi_1 \geq E_{x_1}[F_c(x_1)]$, as r_2 grows or any of p_1, p_2 or r_1 decreases, the optimal value of b_2 (the value at which $\psi(b_2) = 0$) increases in a close neighborhood (to be more exact, we should say "does not decrease"). We cannot extend this statement form "a close neighborhood" to the whole domain of b_2, because $[-F_c(b_2) + E_{x_1}[F_c(x_1 + b_2)]]$ is generally neither increasing nor decreasing.

Regarding (16), in the case studied in theorem 2, if $E(c)$ increases to $E(c)+\delta$ while the variance as well as the shape of density function of c remains unchanged, then the optimal value of b_2 rises exactly by δ.

Here, when PDF of x_1 is shifted ahead, that is when the demand of first group customers increases, $E_{x_1}[F_c(x_1 + b_2)]$ increases (does not decrease) as well. Consequently, the value of the root of $\psi(b_2) = 0$ (optimal b_2), is reduced (does not rise).

This can be summarized as follows. In optimal solution, a positive booking limit is set for the second group (frequent customers) if the ratio of its price rate to expense paid back to a first group customer (price rate plus penalty in the case of cancellation), is high enough. As long as the price rate for frequent customers grows up or the price rate of occasional customers or penalty rate for the customers of either first or second group reduces, the optimal booking limit of regular customers tends to rise and does not decline. Extension of capacity results in higher booking limit of frequent customers. If proportion of price rate to penalty rate for occasional customers is too low, then no capacity is assigned to them. Growth of capacity has an increasing effect on booking limit of occasional customers, while the rise of demand of frequent customers has a decreasing influence on it. More detailed sensitivity analysis is possible, only if the PDF's of demand and capacity are known.

CONCLUSION

In this paper, we formulated a seat inventory control problem for two groups of early-reserving and late-purchasing customers, who pay different prices for the same ticket. We developed a method for a stochastic capacity allocation problem by applying the concept of revenue management, and managed to introduce a new approach. The objective function is unimodal and thus the optimal solution can be found among two possible points, depending on such factors as demand and capacity probability distribution functions, ticket price and cancellation penalty rate for each group.

It is implied from the properties of the optimal solution that as long as the cancellation penalty of group 1 rises or the ticket price of group 2 reduces, the booking limit of group 2 tends to be tighter and at its extreme, it will be zero. That is no reservation for group 2 (early-reserving) should be made when the cancellation penalty of group 1 (late-purchasing) is too high or the price rate of group 2 is too low. Similarly, as long as capacity is more constraining for demands of group 1, booking limit of group 2 will be lower and more limiting.

For further research, it is recommended to find exact formulas for optimal solution, in accordance with some specific probability distribution functions, for both demand and capacity. More specifically, focusing on normal and uniform distributions, which are the most applicable ones in real cases, may lead to more practical results. Another useful way of extending this model is to increase the number of groups. Developing a dynamic request acceptance/rejection policy can be a good idea too. Studying sensitivity of the optimal solution to parameters of the model, such as price rates and penalties, while considering special probability distributions for demand and capacity, is also suggested. Finding interactive policies for environments with more than one competing airlines will be useful too.

Further studies are also needed regarding the situation where plane's capacity is stochastic. Currently, such studies are limited to "Shifting Capacity" or "Moving Curtain" cases. There are lots of works focusing on probability distributions of demand, no-shows, go-shows, etc. However, one can hardly find sources about estimation of capacity probability distribution. Therefore, additional research in this field is also recommended.

APPENDIX

Appendix A : Derivatives of E(R)

To have derivatives of the objective function with respect to decision variable b_2, first we have to find the derivatives of expected values of a_1, a_2, d_1 and d_2 with respect to the mentioned variable:

$$\frac{\partial E(a_1)}{\partial b_2} = 0 \tag{17}$$

$$\frac{\partial E(a_2)}{\partial b_2} = \int_0^{b_2} 0\, dx_2 + b_2 f_2(b_2) + \int_{b_2}^{\infty} f_2(x_2) dx_2 - b_2 f_2(b_2) = \bar{F}_2(b_2) \tag{18}$$

$$\frac{\partial E(d_2)}{\partial b_2} = \int_0^{b_2} 0\, dx_2 + \int_0^{b_2} (b_2 - c) f_c(c) f_2(b_2)\, dc + \int_{b_1}^{\infty} \frac{\partial}{\partial b_1}\left[\int_0^{b_2} (b_2 - c) f_c(c) f_2(x_2)\, dc\right] dx_2 - \int_0^{b_2} (b_2 - c) f_c(c) f_2(b_2)\, dc = F_c(b_2)\bar{F}_2(b_2) \tag{19}$$

$$\frac{\partial E(d_1)}{\partial b_2} = \bar{F}_2(b_2) \int_0^{\infty} \int_{b_2}^{b_2+x_1} f_c(c) f_1(x_1)\, dc\, dx_1 = \bar{F}(b_2) E_{x1}[F_c(b_2 + x_1) - F_c(b_2)] \tag{20}$$

Now, it can be concluded that:

$$\frac{\partial E(R)}{\partial b_2} = \bar{F}_2(b_2)\left[r_2 - (\pi_2 - \pi_1) F_c(b_2) - \pi_1 E_{x_1}[F_c(x_1 + b_2)]\right] \tag{21}$$

Appendix B: Studying convexity/concavity of E(R)

The maximal value of $E(R)$ is obtained from setting (11) to zero, if $E(R)$ is concave, or the second-degree derivative of $E(R)$ is always negative. This property is checked as follows:

$$\frac{\partial^2 E(R)}{\partial b_2^2} = -r_2 f_2(b_2) + \pi_1\left[f_c(b_2)\bar{F}_2(b_2) - F_c(b_2) f_2(b_2)\right] + \pi_2[F_c(b_2) f_2(b_2) - f_2(b_2) E_{x_1}[F_c(b_2 + x_1)] + \bar{F}(b_2) E_{x_1}[f_c(b_2 + x_1)]] \tag{22}$$

Considering second-degree derivative, it can be observed that the objective function is not generally convex or concave. Counter example that is presented below, shows $E(R)$ is neither convex nor concave:

Let:

$c \sim U[10, 15]$, $x_1 \sim U[5, 8]$ and $x_2 \sim U[6, 9]$

while $U[\alpha, \beta]$ represents a continuous uniform probability distribution between α and β. At the point $b_2 = 10$, $\frac{\partial^2 E(R)}{\partial b_2^2} = \frac{\pi_2}{5} > 0$ but at the point $b_2 = 4$, $\frac{\partial^2 E(R)}{\partial b_2^2} = -(\frac{r_2}{3} + \frac{\pi_2}{10}) < 0$, when $\pi_2 > r_2 > 0$. It means in two different points, convexity directions of $E(R)$ function are not the same.

Concluding from this example, we can state that, generally the optimal solution cannot be found by differentiating the objective function.

Appendix C: Boundary properties of E(R)

There are certain boundary properties of the objective function and its derivative that do not depend on the type of probability distribution functions. Considering boundary assumptions (1) and (2), we have:

1. When the booking limit is set to zero, expected revenue and first and second degree derivatives of expected revenue are as follows:

$$b_2 = 0 \Rightarrow \begin{cases} E(R) = r_1 E(x_1) - \pi_1 E_{x_1}[\int_0^{x_1} (x_1 - c) f_c(c)\, dc] \\ \frac{\partial E(R)}{\partial b_2} = r_2 - \pi_1 E_{x_1}[F_c(x_1)] \\ \frac{\partial^2 E(R)}{\partial b_2^2} = \pi_2 E_{x_1}[f_c(x_1)] \geq 0 \end{cases} \tag{23}$$

Since the second degree derivative is positive, the objective function is convex at zero.

2. When the value of booking limit is very big, expected revenue is not sensitive to little changes of it:

$$\lim_{b_2 \to \infty} \left(\frac{\partial E(R)}{\partial b_2} \right) = 0 \tag{24}$$

3. When the booking limit takes large amounts, the objective function will be flat. That is, it is either convex or concave:

$$\lim_{b_2 \to \infty} \frac{\partial^2 E(R)}{\partial b_2^2} = 0 \tag{25}$$

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