



Original Research Article

A Simplified Analytic Approach for Converting Airplane and Cantilever Beams Distributed Loads to Equivalent Concentrated Loads

Reza Khaki^{1*} and Ali Gharibi²

1. Department of Shahid Sattari University of Air Science and Technology, Tehran, Iran

2. Department of Aerospace Engineering, Amirkabir University of Technology, Tehran, Iran

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ABSTRACT

The equations for converting distributed loads to concentrated ones at the arbitrary acting point with equal resulting deflections are specified. These equations are developed for cantilever simple and tapered beams and the same structure that is under various distributed loads on its different parts longitudinally. The equations are established via an analytical process using the resulting formulations from the beam bending integration and the superposition principle. Having obtained enough confidence in the accuracy and validity of the derived equations, the simple and tapered beams are typically assigned. The beams are divided into different parts, and arbitrary and various distributed loads are assigned to each part. The equivalent point loads are calculated for all sections through the suggested equations. The finite element models of the beams are prepared. The distributed and obtained equivalent concentrated loads of each part at the assigned places are applied, and deflections for both loading conditions are achieved separately through the numerical solutions. Finally, the resulting deflections from distributed and equivalent concentrated loading are compared, and the developed equations are validated.

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1 INTRODUCTION

In the analysis of structures, the geometry of the overall structure, material properties, magnitude of loads, and their positions are necessary. During this process, the loads are idealized as point and distributed loads. The point loads are applied at specific points on the structures. Usually, the loads acting on an infinitely small area are called point loads. In terms of analysis, these specific points can

be identified discretely on the structures. The distributed loads are applied along the structures' specified length or area. These loads could be branded as uniformly distributed when the intensity of load for every unit loaded length is the same. Those are called a variable when the load intensity pattern varies in accordance with certain geometrical patterns like triangular, parabolic, and trapezoidal. The loads may vary in an at-random way, or they may be a combination of two or more types of the mentioned loading. In this case, those

* Corresponding Author's Email: reza_khaki@yahoo.com

are distinguished as complex [1]. These loads may be uniform or vary in their distribution. In terms of analysis, the magnitude of the loads and their exertion and the length or area are necessary to calculate the total load applied. Moreover, determining the load's centroid or acting point of the total load must be considered. Thus, researchers have developed methods for calculating the distributed load equivalent to a concentrated load and its acting point in the past.

In 2001, Smith presented a book on the introduction to structural mechanics [2]. He introduced that for evenly distributed uniform loads, the total load can be calculated by multiplying the amount of load per unit of length/area by its exertion length/area. The total load act point or load's centroid for a rectangle is located at half of its length. For the triangular loads, the total load can be achieved by multiplying the maximum intensity of the load per unit of length by half of the loads' exertion length. The acting point of this load is the centroid of the area, which is situated at two-thirds of the length from the apex.

Krenk [3] conducted the mechanics and analysis of beams, columns, and cables. The resulting moments from distributed loads on the supports of beams and columns are considered. He computed the equivalent and acting points of distributed loads using the procedures mentioned in the Ref. [2].

In 2005, the stress effectiveness on the control surfaces, such as ailerons, elevators, and rudder of a pursuit airplane, was considered [4]. The calculation of the moments produced by the loads on the ailerons, elevators, and rudder is attempted. Determination of equivalent load and its centroid for distribution loads on each surface is considered. These are done by dividing each surface into a sufficient number of strips. The center of gravity of the load on each strip is assigned at one-third of its length from the leading edge when the surface is under triangular loading.

From a structural point of view, aircraft wings are usually considered cantilever beams during most structural analyses. For example, a paper on the subject of feasibility modeling of a truss-braced wing as a beam (final) is conducted by Dym and Williams [5]. They presented a case study on the use of elementary beams and truss theory to determine whether a truss-braced wing is structurally feasible. The purpose of this study is solely to see whether a

truss-braced wing is structurally sensible as a prelude to performing a full-scale analysis of such configurations that would include their aerodynamics, materials, controls, and economics. In another research, Gervois and Destarac modeled the aircraft wings as cantilever beams for drag polar invariance with flexibility investigations [6]. Attentive to the aforementioned subjects, all studies about converting the beams' distributed loads to point ones can be helpful in the analysis of air vehicles.

In 2013, Lindeburg studied computation moment from a distributed load [7]. He introduced that the moment of a distributed load, uniform or otherwise, is the product of total force and the distance to the centroid of the distributed load. Thus, the Equation for concentrated load calculation that is equivalent to a distributed load is determined. Moreover, the location of the equivalent load is calculated by another equation. The mentioned equations are developed for all types of distributed loads. By employing the mentioned equations, for a straight beam with a length of L under a uniform transverse loading of W , the equivalent point load and its location are WL and $L/2$, respectively. For the same beam under a triangular distribution, the equivalent point load and its location are $WL/2$ and $2L/3$.

A book on the subject of introduction to numerical and analytical methods with MATLAB for engineers and scientists is presented by Bober [8]. The equivalent resultant force for rectangular and triangular loads and their acting points is achieved by using the mentioned process in Ref. [2]. Furthermore, the beam bending moment and reactions of its supports under rectangular and triangular distributed loads are calculated. Most structural analyses and studies for beams, columns, and aircraft wings were conducted based on calculations of moments and support reaction loads. Thus, for these goals, the abovementioned methods are recognized as sufficient to convert distributed loads to point loads. In some structural investigations and research, concentrating the distributed load on a desired centroid point as a centralized load without changing its effectiveness is necessary. Moreover, some studies may be done to compute structural deflections under distributed loads, such as those performed for beams, columns, and wings. In these researches, converting the distributed to a point load for a specific point must be done as the resulting deflection from both loading conditions becomes equal. This action can lead to

the feasible and precise calculation of aircraft deflections' beams, columns, and wings under various distributed loads on their different parts.

In numerical structural analysis, two standard methods are suggested for transforming the distributed to concentrated loads. The static and work equivalent loads process is based on finite element methods. In the statically equivalent load method, the total distributed loads over one-half of the element are assigned to one nodal point, and those over the other half to another nodal point [9]. In the work-equivalent load method, the work produced by the unknown nodal concentrated loads is set to be equal to the work produced by the actual distributed load. The work-equivalent loads associated with a certain degree of freedom i are obtained by integrating the product of the distributed load function and the i -th shape function over the element length. Such a definition can be generalized to other types of finite elements, such as plates and shells. The method is used to evaluate the body and surface loads at the nodal points.

In 2017, the displacements and internal forces of submerged floating tunnels under hydrodynamic and three-dimensional seismic excitations were investigated by Muhammad et al. [10]. Drag and inertia forces are considered distributed loadings during the numerical simulations. They transformed the above forces into equivalent nodal forces by employing the work-equivalence load method. The study on the Combination of experiments and CFD analysis for in-flight wing deflection estimation is presented by Ovesy et al. [11], 2021. In this research, the wing of a typical aircraft is divided into 33 parts, and the total applied loads for each part are determined via the summation of aerodynamic and inertia loads at any flight condition. They converted the total surface/traction to the concentrated point loads through the work-equivalent load method. To this end, each part of the wing is considered as one volume element.

In engineering sciences, especially mechanics and aerospace, it is necessary and urgent to convert the distributed load to a point load in many cases. For this purpose, the work-equivalence method has been used by many researchers in recent years. Maurice et al. [12] by publishing an article on the subject of the application of a three-noded finite element beam model to beam on elastic foundation, and Truong et al. [13] by presenting a paper under the title of a mass-spring fluid-structure interaction solver: application to flexible revolving wings, are

among the mentioned researchers. Furthermore, the work-equivalence method is employed by Mao et al. [14] in the paper on the subject of an electromagnetic load identification method based on the polynomial structure selection technique in 2024.

The mentioned methods provide results with reasonable accuracy when sufficient elements are used. For transforming the applied distributed load of each part of the beam or aircraft wings to the concentrated loads at the desired acting point, the corresponding part must be considered as one element. Thus, the precision of the methods mentioned above may be much decreased. It is obvious that applying the distributed load during experimental tests and the conceptual/initial design process is not feasible and precise. Moreover, the employment of finite elements for the work-equivalent load method during the mentioned process is not timely. Therefore, Attentive to the aforementioned objects, it is evident that any endeavor to convert the distributed loads to concentrated loads with the desired centroid point seems worthwhile.

The current paper develops the equations for converting a distributed load to a concentrated load at an arbitrary acting point. These equations are determined for cantilever simple and tapered beams and the same structure that is under various distributed loads on its different parts longitudinally. The equations are established via an analytical process by using the resulting formulations from the integration of beam bending and the principle of superposition methods.

Moreover, for the validity of the developed equations, the simple and tapered beams are typically assigned. The beams are divided into different parts, with arbitrary and various distributed loads applied to each part. The equivalent point loads are calculated for all sections through the latter equations. The finite element models of the beams are prepared. The distributed and equivalent concentrated loads of each part at the assigned places are applied, and deflections are determined for both loading conditions separately through numerical solutions. Finally, the resulting deflections from distributed and equivalent concentrated loading are compared, and the developed equations are validated.

It is noted that the resulting equations are quite efficient and useful for converting the distributed load to a point load. The presented analytical method is very costly and time-consuming but feasible

during experimental tests and the conceptual/initial design stage.

1.1 Analytical procedures for the determination of converter equations

Figure 1 shows a typical geometry of a simple cantilever beam with constant height and width that is under a concentrated load P at its free end A .

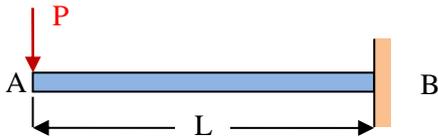


Fig. 1. Example of a simple beam with a concentrated load

The equations below [15] can obtain the maximum deflection and slope at A .

$$y_A = -\frac{PL^3}{3EI} \text{ and } \theta_A = -\frac{PL^2}{2EI} \quad (1)$$

Where P is the magnitude of the point load, y_A is the free end deflection, θ_A is the slope at A and L , E and I are the length, elastic modulus, and moment of inertia, respectively.

The beam mentioned above is subjected to a uniformly distributed load of intensity w , as shown in Figure 2.

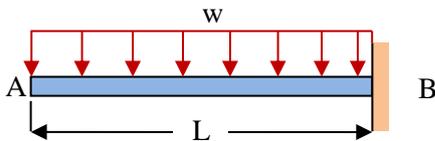


Fig. 2. Example of a simple beam with a uniformly distributed load

At the free end A , the beam will have the maximum deflection and slope, which the equations below [11] can determine.

$$\delta_A = -\frac{wL^4}{8EI} \text{ and } \phi_A = -\frac{wL^3}{6EI} \quad (2)$$

δ_A and ϕ_A are the deflection and slope at the free end A .

Converting the applied distributed load to the beam in Figure 2, which is under uniform load longitudinally to a point load with regard to equal deflection, is facile. The equivalent concentrated

load of this loading condition at A with equal deflection is determined as follows.

$$\begin{aligned} &\text{Resulted in deflection via distributed load } (\delta_A) \\ &= \\ &\text{Resulted deflection via equivalent point load } (y_A) \\ \Rightarrow &-\frac{wL^4}{8EI} = -\frac{P_1L^3}{3EI} \\ \Rightarrow &P_1 = \frac{3wL}{8EI} = \text{equivalent point load} \end{aligned} \quad (3)$$

For example, a beam under application of various intensities of distributed load is depicted in Figure 3. Calculating equivalent point loads for its parts to make a maximum deflection the same as the main loads is not feasible.

Converting the loads w_0 , w_1 and w_2 to concentrated load at the centroid points B , C , and D are considered, respectively

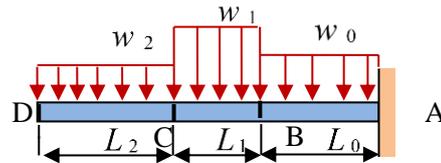


Fig. 3. Example of a simple beam with various distributed loads

Using Equation (3), the equivalent loads for this loading condition at the mentioned point will be calculated as follows.

$$P_B = \frac{3w_0L_0}{8EI}, \quad P_C = \frac{3w_1L_1}{8EI}, \quad P_D = \frac{3w_2L_2}{8EI} \quad (4)$$

Where P_B , P_C and P_D are the equivalent concentrated loads for w_0 , w_1 and w_2 at the points B , C , D , respectively.

$$(\delta_C)_a = \frac{w(L_1+L_2)^4}{8EI} \text{ and } (\theta_C)_a = \frac{w(L_1+L_2)^3}{6EI} \quad (5)$$

As observed in Equation (4), for converting w_0 effectiveness of w_1 and w_2 on the deflection of point B is not considered. The effects of w_0 and w_2 on the deviation of point C is ignored as the effects of w_0 and w_1 on the point D deflections while converting w_1 and w_2 . Whereas each of the mentioned loadings has a role in the deviations of other parts. Because of this, the conversion action by employing Equation (3) is not correct for a

beam under the application of the various intensities of distributed load. In the current paper, the formulations for converting various distribution loads to concentrated loads at arbitrary points with the same resulting deflection are considered. To this end, a simple cantilever beam with three parts under a uniformly distributed load is considered, as shown in Figure 4.

The superposition principle is employed to compute deflection at the free end D. According to this origin, the loading condition in the mentioned figure is substituted with two conditions in Figure 5 [15].

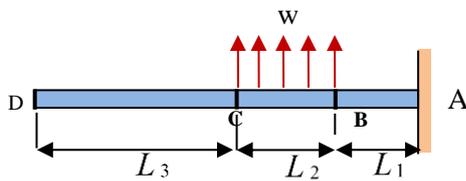


Fig. 4. Simple cantilever beam divided into three parts.

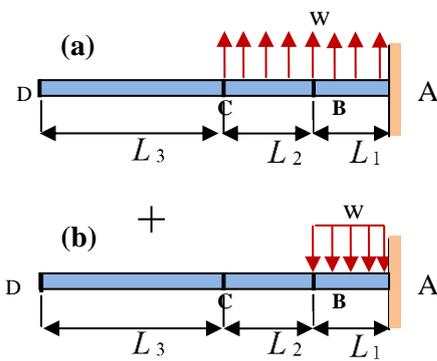


Fig. 5. Substituted conditions for Figure 4 load case

The deviation of free end D will be achieved through the summation of the resulting deflections from load conditions a and b. For each of the loadings a and b, the slope and deflection at C and, as a result, at D will be determined by using Equations (2) as follows:

Loading condition (a):

On the other hand:

$$\begin{aligned}
 (\delta_D)_a &= (\delta_C)_a + (\theta_C)_a \times L_3 \\
 &= \frac{w(L_1+L_2)^4}{8EI} + \frac{w(L_1+L_2)^3}{6EI} \times L_3
 \end{aligned} \quad (6)$$

Loading condition (b):

$$(\delta_B)_b = -\frac{wL_1^4}{8EI} \text{ and } (\theta_B)_b = -\frac{wL_1^3}{6EI} \quad (7)$$

In portion BC, and also CD the elastic curve is a straight line, thus:

$$\begin{aligned}
 (\theta_B)_b &= (\theta_C)_b, \\
 (\delta_D)_b &= (\delta_B)_b + (\theta_B)_b \times (L_2 + L_3) \\
 &= -\frac{wL_1^4}{8EI} - \frac{wL_1^3}{6EI} \times (L_2 + L_3)
 \end{aligned} \quad (8)$$

$$\begin{aligned}
 (\delta_D)_w &= (\delta_D)_a + (\delta_D)_b \\
 &= \frac{w(L_1+L_2)^4}{8EI} + \frac{w(L_1+L_2)^3}{6EI} \times L_3 \\
 &\quad - \frac{wL_1^4}{8EI} - \frac{wL_1^3}{6EI} \times (L_2 + L_3)
 \end{aligned} \quad (9)$$

$$\Rightarrow (\delta_D)_w = \frac{w}{EI} \left[\begin{aligned} &(L_1+L_2)^3 \times \left(\frac{L_1+L_2+L_3}{8} + \frac{L_3}{6} \right) \\ &- L_1^3 \times \left(\frac{L_1}{8} + \frac{L_2+L_3}{6} \right) \end{aligned} \right] \quad (10)$$

Where $(\delta_D)_w$ is the total deflection at free end D corresponding to distributed load w.

Now, it is assumed that the equivalent of load w is a concentrated load similar P at the centroid C as the created deflection via both of them at the D is equal. Settling the load P at the centroid point, C is depicted in Figure 6.

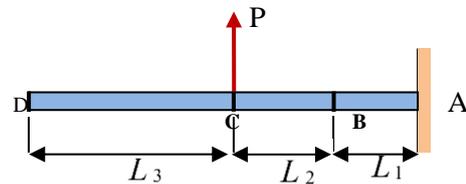


Fig. 6. Settling the substituted point load

For calculating the magnitude of P according to w, the created deflection by P at the free end D should be computed. To this end, the deviation and slope at the C is determined by using Equation (1) as expressed below.

$$(\delta_C)_P = \frac{P(L_1+L_2)^3}{3EI}, (\theta_C)_P = \frac{P(L_1+L_2)^2}{2EI} \quad (11)$$

$(\delta_C)_P$ and $(\theta_C)_P$ are the created deflection and slope at C corresponding to the concentrated load P .

$$\begin{aligned} (\delta_D)_P &= (\delta_C)_P + (\theta_C)_P \times L_3 \\ &= \frac{P(L_1+L_2)^3}{3EI} + \frac{P(L_1+L_2)^2}{2EI} \times L_3 \end{aligned} \quad (12)$$

$$\Rightarrow (\delta_D)_P = \frac{P}{EI} \left[(L_1+L_2)^2 \times \left(\frac{L_1+L_2}{3} + \frac{L_3}{2} \right) \right] \quad (13)$$

Where $(\delta_D)_P$ is the deflection at the free end D corresponding to the load P .

Since finding the equivalent of w that generates equal deflection is considered, thus Equations (10) and (13) are quitted together.

$$(\delta_D)_W = (\delta_D)_P \quad (14)$$

$$\begin{aligned} \Rightarrow \frac{w}{EI} \left[(L_1+L_2)^3 \times \left(\frac{L_1+L_2}{8} + \frac{L_3}{6} \right) - L_1^3 \times \left(\frac{L_1}{8} + \frac{L_2+L_3}{6} \right) \right] \\ = \frac{P}{EI} \left[(L_1+L_2)^2 \times \left(\frac{L_1+L_2}{3} + \frac{L_3}{2} \right) \right] \end{aligned} \quad (15)$$

Finally, the equation for calculating the equivalent of the distributed load on the simple cantilever beams is achieved and written below.

$$P = \frac{w \times \left[(L_1+L_2)^3 \times \left(\frac{L_1+L_2}{8} + \frac{L_3}{6} \right) - L_1^3 \times \left(\frac{L_1}{8} + \frac{L_2+L_3}{6} \right) \right]}{\left[(L_1+L_2)^2 \times \left(\frac{L_1+L_2}{3} + \frac{L_3}{2} \right) \right]} \quad (16)$$

In the mentioned Equation L_1 is the distance from the support up to the beginning of the loading strip and L_3 is the distance from the end of the loading part up to the end of the beam, and the L_2 is length of the loading strip. As an example, for loading conditions in Figure 2, L_1 and L_3 are equal to zero, thus Equation (16) will be changed to Equation (3).

In Equation (16), the effects of applied load on the sections that are established before and after the loading part are seen.

Moreover, in this section of the current paper, determining an equation for converting the various distributed loads of a tapered cantilever beam with various widths to the concentrated loads is

performed. Therefore, all the procedures mentioned for simple beams will be employed.

Figure 7 shows a tapered cantilever beam that is under a distributed load w at the part BC . The cross sections of the beam are rectangular with various widths and depths. The depth and width at the support A are t_A , b_A and at the free end t_D , b_D , respectively.

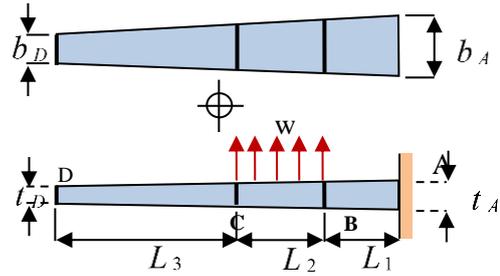


Fig. 7. Typical tapered cantilever beam divided into three parts

Like the simple beam process, a typical tapered cantilever beam is substituted with two cases a and b . The deflection of free end D will be achieved by summing, resulting in deflections from these load conditions. For computing of deflection according to loading conditions a and b the Equations (6) and (8) is expressed as below, respectively.

$$(\delta_D)_a = \frac{w(L_1+L_2)^4}{8E \bar{I}_{1,2}} + \frac{w(L_1+L_2)^3}{6E \bar{I}_{1,2}} \times L_3 \quad (17)$$

$$(\delta_D)_b = -\frac{w L_1^4}{8E I_1} - \frac{w L_1^3}{6E I_1} \times (L_2+L_3) \quad (18)$$

Where E is constant for the beam and $\bar{I}_{1,2}$, $\bar{I}_{1,2}$ are the average inertia moment of part 1 and parts 1,2 together, respectively. Total deflection at free end D corresponding to distributed load w for tapered cantilever beams can be achieved from the Equation below.

$$\begin{aligned} (\delta_D)_w &= (\delta_D)_a + (\delta_D)_b \\ &= \frac{w(L_1+L_2)^4}{8E \bar{I}_{1,2}} + \frac{w(L_1+L_2)^3}{6E \bar{I}_{1,2}} \times L_3 \\ &\quad - \frac{w L_1^4}{8E I_1} - \frac{w L_1^3}{6E I_1} \times (L_2+L_3) \end{aligned} \quad (19)$$

$$\Rightarrow (\delta_D)_w = \frac{w}{E} \left[\frac{(L_1+L_2)^3}{I_{1,2}} \times \left(\frac{L_1+L_2}{8} + \frac{L_3}{6} \right) - \frac{L_1^3}{I_1} \times \left(\frac{L_1}{8} + \frac{L_2+L_3}{6} \right) \right] \quad (20)$$

As observed, Equations (20) and (10) are similar, but the inertia moments in 20 are not constant.

Figure 8 shows a concentrated load as an equivalent of w that is placed at the acting point C . It is assumed that the resulting deviations at the free end D corresponding to w and P are equal.

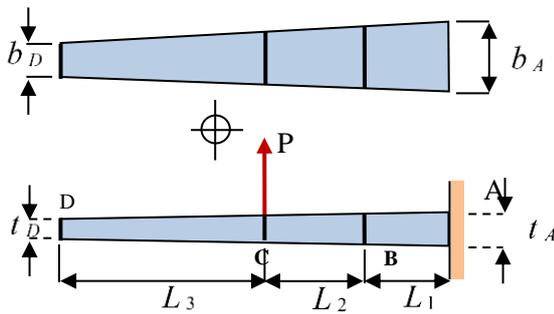


Fig. 8. Arrangement of substituted concentrated load

The created deflection due to P at the free end D is calculated by the process that is used for a simple beam. Because of this, Equation (12) is changed to Equation (21) for tapered cantilever beams with various widths.

$$(\delta_D)_P = \frac{P(L_1+L_2)^3}{3E I_{1,2}} + \frac{P(L_1+L_2)^2}{2E I_{1,2}} \times L_3 \quad (21)$$

$$\Rightarrow (\delta_D)_P = \frac{P}{E I_{1,2}} \left[(L_1+L_2)^2 \times \left(\frac{L_1+L_2}{3} + \frac{L_3}{2} \right) \right] \quad (22)$$

The magnitude of P can be equivalent to w if the resulted deflection via these loading conditions at D be equal. Thus, the Equations (20) and (22) are quitted with together.

$$(\delta_D)_W = (\delta_D)_P \quad (23)$$

$$\Rightarrow \frac{w}{E} \left[\frac{(L_1+L_2)^3}{I_{1,2}} \times \left(\frac{L_1+L_2}{8} + \frac{L_3}{6} \right) - \frac{L_1^3}{I_1} \times \left(\frac{L_1}{8} + \frac{L_2+L_3}{6} \right) \right] = \frac{P}{E I_{1,2}} \left[(L_1+L_2)^2 \times \left(\frac{L_1+L_2}{3} + \frac{L_3}{2} \right) \right] \quad (24)$$

$$\Rightarrow P = \frac{W \times \left[\frac{(L_1+L_2)^3}{I_{1,2}} \times \left(\frac{L_1+L_2}{8} + \frac{L_3}{6} \right) \right]}{\left[\frac{(L_1+L_2)^2}{I_{1,2}} \times \left(\frac{L_1+L_2}{3} + \frac{L_3}{2} \right) \right]} - \frac{W \times \left[\frac{L_1^3}{I_1} \times \left(\frac{L_1}{8} + \frac{L_2+L_3}{6} \right) \right]}{\left[\frac{(L_1+L_2)^2}{I_{1,2}} \times \left(\frac{L_1+L_2}{3} + \frac{L_3}{2} \right) \right]} \quad (25)$$

If the moment of inertia was constant longitudinally, the terms $\overline{I_{1,2}}$ and $\overline{I_1}$ can be deleted from the numerator and denominator, and Equation (25) becomes Equation (16).

Prediction of $\overline{I_{1,2}}$ and $\overline{I_1}$ is an essential problem in Equation (25). Some of the researchers, such as Shukla [17], developed the methods for inertia moment estimation. The aim of the current paper is to find a feasible method for the prediction of $\overline{I_{1,2}}$ and $\overline{I_1}$ as the error of the outcomes from Equation (25) becomes minimized. The upward view of Figure 7 and its areas is depicted in Figure 9.

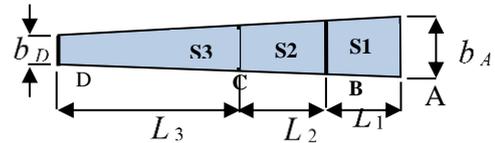


Fig. 9. Overview of Figure 7 and area presentation.

The parameters b_A , b_B , b_C and b_D are the width of the beam at A , B , C , and D sections. For the beams with linear width changes related to length, the average width at a distance x from support can be achieved through the equations below.

$$\begin{aligned} \overline{b} &= \frac{x}{L} \times \frac{(b_D - b_A)}{2} + b_A, \\ \overline{b}_1 &= \frac{L_1}{L} \times \frac{(b_D - b_A)}{2} + b_A, \\ \overline{b}_2 &= \frac{(L_1 + L_2/2)}{L} \times (b_D - b_A) + b_A \end{aligned} \quad (26)$$

Where \overline{b} is the average width at the position x from the support, and \overline{b}_1 , \overline{b}_2 are the average widths of areas 1 and 2, respectively.

As observed in Figure 7, the w is a longitudinally distributed load and applied at the L_2 that is the length of area 2. This load is divided by the average width of area 2 and converted to the surface expanded load. Thus:

$$q = \frac{w}{b_2} \quad (27)$$

Substituting q in to the Equations (17) and (18) will lead to changing those to below equations.

$$(\delta_D)_a = \frac{q \times \overline{b_{1,2}} \times (L_1 + L_2)^4}{8E \overline{I_{1,2}}} + \frac{q \times \overline{b_{1,2}} (L_1 + L_2)^3}{6E \overline{I_{1,2}}} \times L_3 \quad (28)$$

$$(\delta_D)_b = -\frac{q \times \overline{b_1} \times L_1^4}{8E \overline{I_1}} - \frac{q \times \overline{b_1} \times L_1^3}{6E \overline{I_1}} \times (L_2 + L_3) \quad (29)$$

$$(\delta_D)_w = (\delta_D)_a + (\delta_D)_b = \frac{q}{E} \left[\frac{\overline{b_{1,2}} \times (L_1 + L_2)^3}{\overline{I_{1,2}}} \times \left(\frac{L_1 + L_2}{8} + \frac{L_3}{6} \right) - \frac{\overline{b_1} \times L_1^3}{\overline{I_1}} \times \left(\frac{L_1}{8} + \frac{L_2 + L_3}{6} \right) \right] \quad (30)$$

$$(\delta_D)_W = (\delta_D)_P \quad (31)$$

In the above equations $\overline{b_{1,2}}$, the parameter is the average width of areas 1 and 2 that is generated from areas 1 and 2 summations. Further, the Equation (25) is converted to Equation (32) by equalizing the Equations (22) and (30).

$$\Rightarrow P = \frac{q \times \left[\frac{\overline{b_{1,2}} \times (L_1 + L_2)^3}{\overline{I_{1,2}}} \times \left(\frac{L_1 + L_2}{8} + \frac{L_3}{6} \right) \right]}{\left[\frac{(L_1 + L_2)^2}{\overline{I_{1,2}}} \times \left(\frac{L_1 + L_2}{3} + \frac{L_3}{2} \right) \right]} - \frac{q \times \left[\frac{\overline{b_1} \times L_1^3}{\overline{I_1}} \times \left(\frac{L_1}{8} + \frac{L_2 + L_3}{6} \right) \right]}{\left[\frac{(L_1 + L_2)^2}{\overline{I_{1,2}}} \times \left(\frac{L_1 + L_2}{3} + \frac{L_3}{2} \right) \right]} \quad (32)$$

For the beams with rectangular cross-sections, the inertia moment can be achieved by the below formula.

$$I = \frac{b \times h^3}{12} \Rightarrow \overline{I_1} = \frac{\overline{b_1} \times \overline{h_1^3}}{12} \quad (33)$$

And $\overline{I_{1,2}} = \frac{\overline{b_{1,2}} \times \overline{h_{1,2}^3}}{12}$

By replacing Equation (33) into Equation (32):

$$\Rightarrow P = \frac{q \times \left[\frac{(L_1 + L_2)^3}{\overline{h_{1,2}^3}} \times \left(\frac{L_1 + L_2}{8} + \frac{L_3}{6} \right) \right]}{\left[\frac{(L_1 + L_2)^2}{\overline{b_{1,2}} \times \overline{h_{1,2}^3}} \times \left(\frac{L_1 + L_2}{3} + \frac{L_3}{2} \right) \right]} - \frac{q \times \left[\frac{L_1^3}{\overline{h_1^3}} \times \left(\frac{L_1}{8} + \frac{L_2 + L_3}{6} \right) \right]}{\left[\frac{(L_1 + L_2)^2}{\overline{b_{1,2}} \times \overline{h_{1,2}^3}} \times \left(\frac{L_1 + L_2}{3} + \frac{L_3}{2} \right) \right]} \quad (34)$$

The difference between the magnitudes of $\overline{h_{1,2}^3}$ and $\overline{h_1^3}$ is less. This diversity is smaller when the strip L_2 is small or its distance from the support (L_1) is greater. Therefore, it is assumed that the magnitude of $\overline{h_1^3}$ is equal to $\overline{h_{1,2}^3}$, and by this assumption, Equation (34) is expressed below.

$$P = \frac{q \times \left[(L_1 + L_2)^3 \times \left(\frac{L_1 + L_2}{8} + \frac{L_3}{6} \right) \right]}{\left[\frac{(L_1 + L_2)^2}{\overline{b_{1,2}}} \times \left(\frac{L_1 + L_2}{3} + \frac{L_3}{2} \right) \right]} - \frac{q \times \left[L_1^3 \times \left(\frac{L_1}{8} + \frac{L_2 + L_3}{6} \right) \right]}{\left[\frac{(L_1 + L_2)^2}{\overline{b_{1,2}}} \times \left(\frac{L_1 + L_2}{3} + \frac{L_3}{2} \right) \right]} \quad (35)$$

Where

$$\overline{b_{1,2}} = \frac{(L_1 + L_2)}{L} \times (b_D - b_A) + b_A \quad (36)$$

Equation (25) is sufficient for any tapered cantilever beam. Equation (35) is the particular form of Equation (25) that is sufficient for tapered cantilever beams with variable width and rectangular cross sections.

In the following sections of the current paper, for some examples, the finite element models are prepared. The distributed loads are converted to concentrated loads by using Equations (16) and

(35). The distributed loads and corresponding point loads are applied to the models, and the obtained deflections from the two processes are compared.

2 FINITE ELEMENT MODELING

Three-dimensional finite element analyses are considered to investigate the accuracy of the obtained equations for converting the various distributed loads to concentrated ones. To this end, some arbitrary geometry with desired properties of simple and tapered beams is designed. Macro programs are developed using the ANSYS Parametric Design Language (APDL) capability to handle the modeling, mesh generation, loading, and linear solutions process. In these analyses, isotropic 8-node solid elements are used to model the simple and tapered cantilever beams. Some of the geometries and meshes of the under investigation beams (the beams in question) are typically shown in Figures 10 and 11.

Totally the major steps of the Macro programs are as follows:

- (i) Geometry and finite element mesh generation of the problem.
- (ii) Defining the loading, constraints, and material properties.
- (iii) Performing the linear elastic solution.
- (iv) Listing the node deflection.

The abovementioned process is conducted twice for all examples, once for the distributed loads condition and again for equivalent concentrated loading. It is noted that the procedures for distributed and point load conditions are the same, just defining the loading should be changed. After each solution, the resulting deflections via applying the distributed and equivalent point loads are recorded and compared for each example.

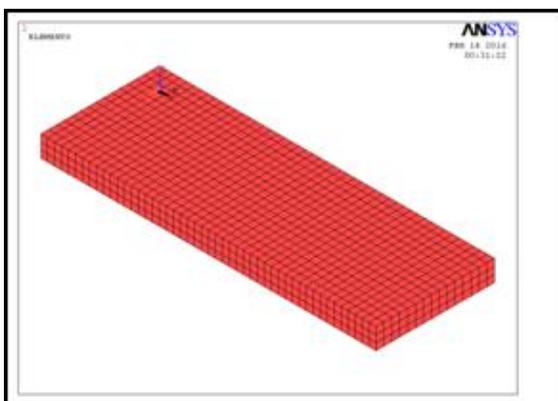


Fig. 10. Typical 3-D finite element model and mesh of simple beams

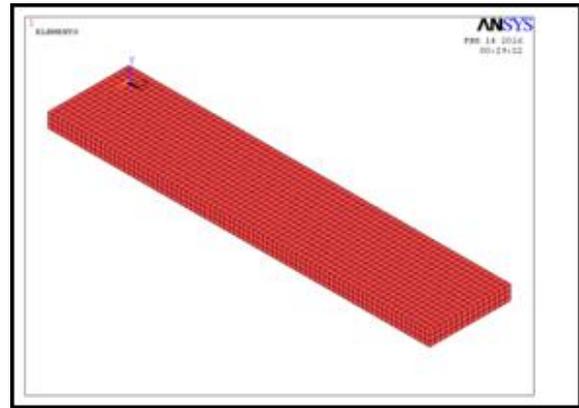


Fig. 10. Typical 3-D finite element model and mesh of simple beams

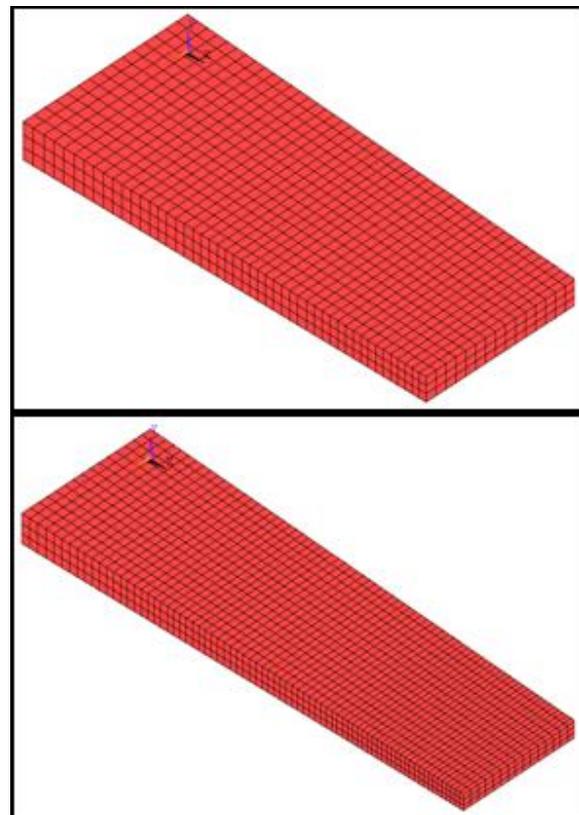


Fig. 11. Typical 3-D finite element model and mesh of tapered and variable-width beams

3 RESULTS AND DISCUSSIONS

The equations for transforming the distributed loads to concentrated loads at the desired acting points based on an analytical process are developed. It is emphasized that the resulting deflections via equivalent loads must agree with those from actual distributed loads. Thus, the validation of the derived equations is attempted by following the steps outlined below. The examples of simple and tapered beams with scaled dimensions the same as aircraft

wings and subjected to various arbitrarily distributed loads are typically chosen. Converting the distributed loads to concentrated ones is performed through the developed equations, and the magnitudes of their equivalent are obtained.

The finite element models for all of the examples are prepared using the ANSYS software, as mentioned before. The actual distributed and corresponding equivalent loads of each sample that are achieved via derived equations are applied to the relative models separately. Finally, the obtained deflection distributions via the aforementioned procedures are compared.

Figure 12 shows the simple beam with the optional properties, thickness, and width, and with a length of 1000 mm under various distributed loads at the different parts. Determination of equivalent loads for strips 1 to 5 at points *B* to *F* through Equation (16) is expected respectively. The

magnitudes of used parameters in this Equation are recorded in Table 1.

Table 1. Typical magnitudes of used variables in Equation (16)

Strip No	W (N/mm)	L ₁ (mm)	L ₂ (mm)	L ₃ (mm)
1	100	0	200	800
2	50	200	125	675
3	125	325	150	525
4	200	475	225	300
5	275	700	300	0

Substituting the mentioned parameters in Equation (16), this Equation for strips 1 and 2 is expressed below.

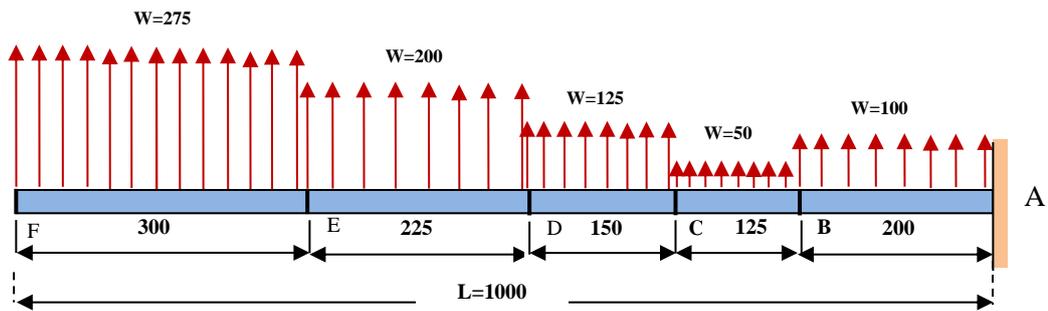


Fig. 12. A Typical simple beam subjected to various distributed loads

$$P_B = \frac{100 \times \left[(0+200)^3 \times \left(\frac{0+200}{8} + \frac{800}{6} \right) \right]}{\left[(0+200)^2 \times \left(\frac{0+200}{3} + \frac{800}{2} \right) \right]}$$

$$\frac{100 \times \left[0^3 \times \left(\frac{0}{8} + \frac{200+800}{6} \right) \right]}{\left[(0+200)^2 \times \left(\frac{0+200}{3} + \frac{800}{2} \right) \right]} = 6785.7 \text{ N}$$

$$P_C = \frac{50 \times \left[(200+125)^3 \times \left(\frac{200+125}{8} + \frac{675}{6} \right) \right]}{\left[(200+125)^2 \times \left(\frac{200+125}{3} + \frac{675}{2} \right) \right]}$$

$$\frac{50 \times \left[200^3 \times \left(\frac{200}{8} + \frac{125+675}{6} \right) \right]}{\left[(200+125)^2 \times \left(\frac{200+125}{3} + \frac{675}{2} \right) \right]} = 4236.28 \text{ N}$$

(37)

The equivalent concentrated loads for points D, E, and F are calculated by using the same procedure. Table 2 shows the transformed distributed loads of the sample beam that are calculated via the derived Equation.

Table 2. Transformed distributed loads at points *B* up to *F*

Magnitude of equivalent point loads				
<i>P_B</i>	<i>P_C</i>	<i>P_D</i>	<i>P_E</i>	<i>P_F</i>
6785.7	4236.3	13802	33456.9	64215.9

In the next step, the actual distributed loads and their equivalent are applied to the FEM model, as shown in Figure 13, separately. The numerical solutions are executed, and the magnitudes of beam deflection are achieved for the two aforementioned loading conditions.

The obtained deflections due to applying distributed and equivalent point loads are depicted in Figure 14, and the corresponding beam deflections distributions are compared in Figure 15.

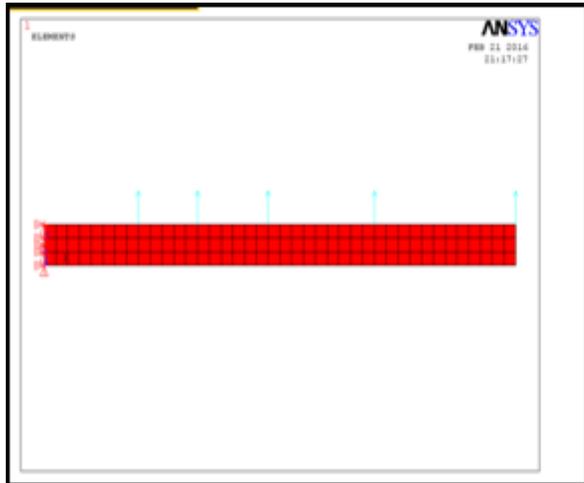
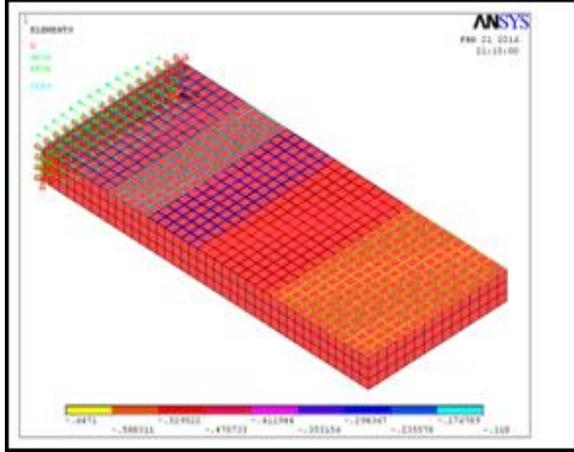


Fig. 13. Sample beam model subjected to various distributed and equivalent point loads

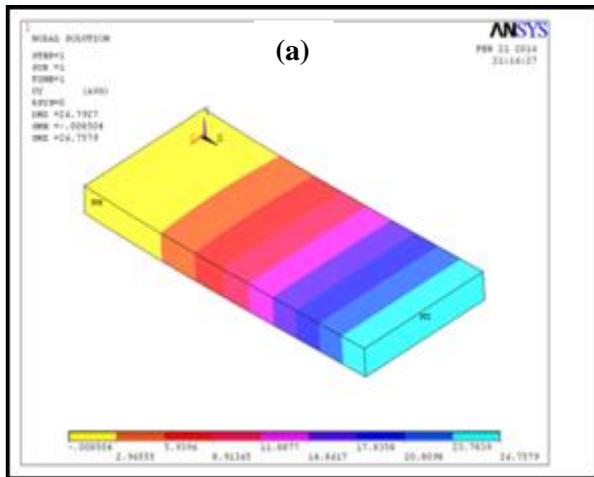


Fig.14. Resulted deflections due to applying (a) distributed and (b) equivalent point loads.

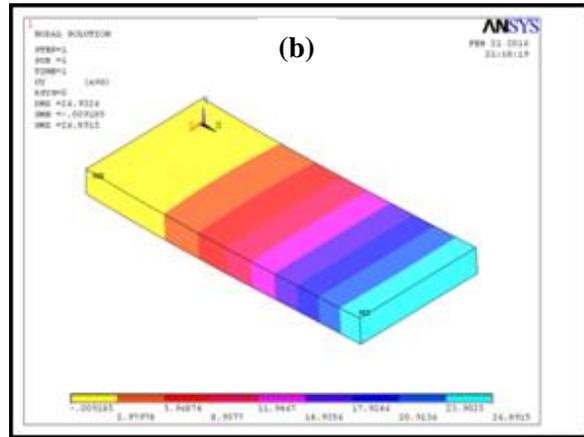


Fig. 14. Resulted deflections due to applying (a) distributed and (b) equivalent point loads.

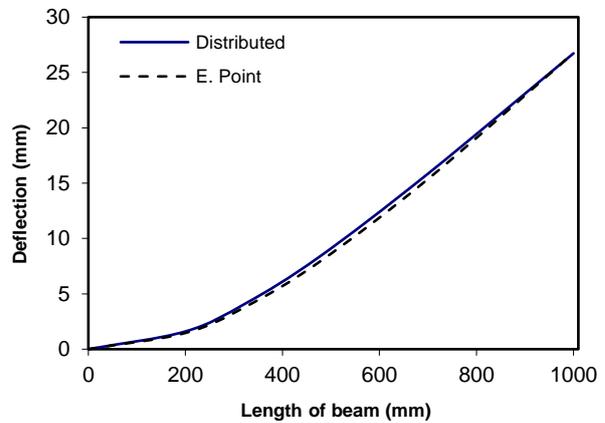


Fig. 15. Comparison of beam deflection distribution by applying distributed and equivalent point loads

For another example, a simple cantilever beam with constant optional properties, thickness, and width is considered. The length of the beam is 1700 mm and is divided into 10 unequal parts, which are subjected to various distributed loads according to Table 3. The equivalent concentrated loads for each strip are achieved by employing Equation (16) and are filed in Table 3.

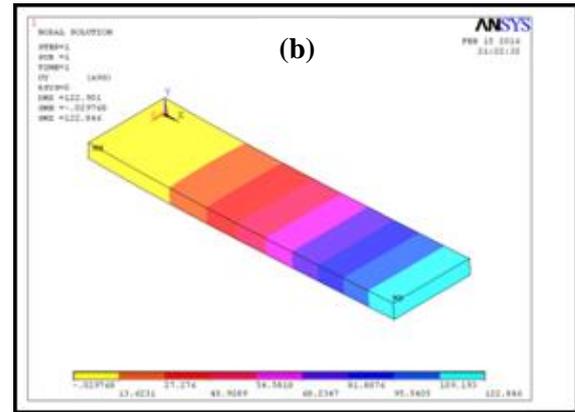
Table 3. Typical data for a beam with a length of 1700 mm

Strip No	W (N/mm)	L_1 (mm)	L_2 (mm)	L_3 (mm)	P (N)
1	50	0	100	1600	1675.0
2	200	100	250	1350	23187.4
3	150	350	200	1150	20796.5
4	175	550	150	1000	21356.7
5	100	700	200	800	16231.6
6	225	900	300	500	53843.1

Strip No	W (N/mm)	L_1 (mm)	L_2 (mm)	L_3 (mm)	P (N)
7	25	1200	50	450	1208.4
8	75	1250	100	350	7048.6
9	125	1350	150	200	17278.0
10	100	1500	200	0	18239.4

The same as the preview example, the actual distributed and equivalent point loads are applied to the FEM model of the current beam separately. The beam deflection magnitudes and the corresponding beam deflection distributions are obtained when the numerical solutions are accomplished. Figure 16 shows the deduced deflections due to applying aforesaid loading conditions, as Figure 17 shows the corresponding beam deflection distributions.

Figures 14 to 17 show the best agreements between beam deflection magnitudes and distributions that are generated by applying the actual distributed loads and their equivalent to the FEM model. Thus, it can be concluded that the derived Equation (16) is perfect for converting the distributed to concentrated loads.



Continuation of Fig. 16. Resulted deflections for a beam with a length of 1700 mm by applying (a) distributed and (b) equivalent point loads.

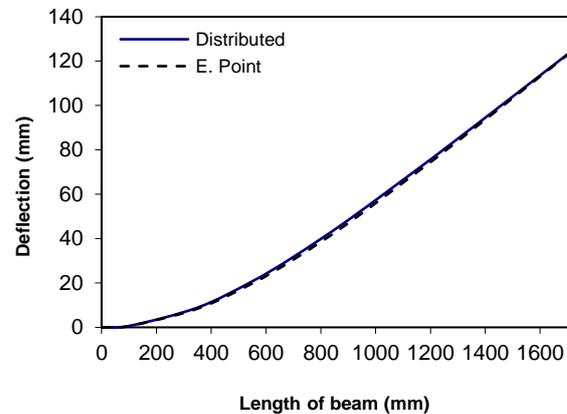


Fig. 17. Comparison of beam deflection distribution by applying distributed and equivalent point loads (for a beam with a length of 1700 mm)

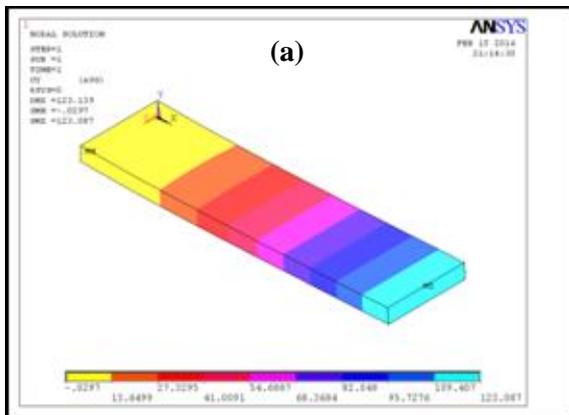


Fig. 16. Resulted deflections for a beam with a length of 1700 mm by applying (a) distributed and (b) equivalent point loads.

Figure 18 shows a tapered cantilever beam AE with the optional properties that are applied to various distributed loads at different parts. The cross sections of the beam are rectangular with widths and depths $b_A t_A$ at support A and $b_E t_E$ at a free end.

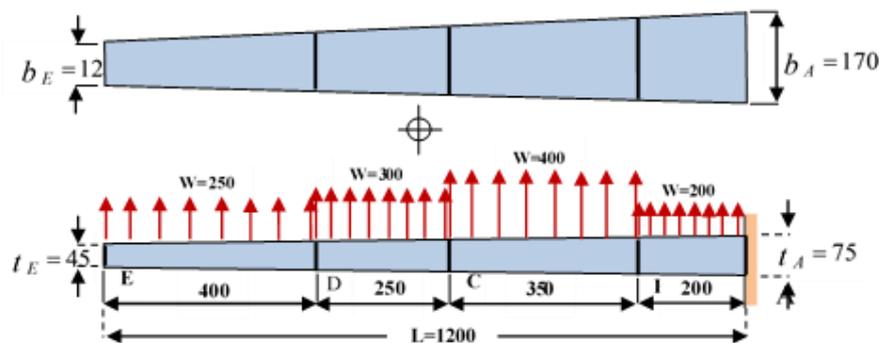


Fig. 18. Typical tapered cantilever beam subjected to various distributed loads

The applied distributed loads to strips 1 to 4 are converted to concentrated loads at points *B* to *E* by employing the derived Equation (35), respectively. To this end, the magnitudes of variables used in this Equation are recorded in Table 4.

Table 4. Typical magnitudes of used variables in Equation (35)

Strip No	W (N/mm)	q (N/mm ²)	L ₁ (mm)	L ₂ (mm)	L ₃ (mm)
1	200	1.20471	0	200	1000
2	400	2.580455	200	350	650
3	300	2.097726	550	250	400
4	250	1.922929	800	400	0

The parameter $\overline{b_{1,2}}$ is calculated for each part through Equation (36) as follows.

For strip 1:

$$\begin{aligned} \overline{b_{1,2}} &= \frac{(L_1 + L_2)}{L} \times (b_E - b_A) + b_A \\ &= \frac{(0 + 200)}{1200} \times (122 - 170) + 170 = 162 \text{ mm} \end{aligned}$$

And for strip 2:

$$\begin{aligned} \overline{b_{1,2}} &= \frac{(L_1 + L_2)}{L} \times (b_E - b_A) + b_A \\ &= \frac{(200 + 350)}{1200} \times (122 - 170) + 170 = 148 \text{ mm} \end{aligned}$$

The calculated magnitudes $\overline{b_{1,2}}$ for strips 3 and 4 are 138mm and 122mm, respectively.

As an example, Equation (35) for equivalent point load determination of strip 1 is expressed as follows.

$$\begin{aligned} P_B &= \frac{1.20471 \times \left[(0+200)^3 \times \left(\frac{0+200}{8} + \frac{1000}{6} \right) \right]}{\left[\frac{(0+200)^2}{162} \times \left(\frac{0+200}{3} + \frac{1000}{2} \right) \right]} \\ &= \frac{1.20471 \times \left[0^3 \times \left(\frac{0}{8} + \frac{200+1000}{6} \right) \right]}{\left[\frac{(0+200)^2}{162} \times \left(\frac{0+200}{3} + \frac{1000}{2} \right) \right]} \\ &= \frac{1.20471 \times 200^3 \times \frac{9200}{48}}{\frac{200^2 \times 3400}{162 \times 6}} = 13202.2 \text{ N} \end{aligned}$$

The same process is conducted for calculating the equivalent concentrated loads of strips 2, 3, and 4 at points C, D, and E, and transformed loads are filed in Table 5.

Table 5. Transformed distributed loads at points *B* up to *E*.

Magnitude of equivalent point loads			
<i>P_B</i>	<i>P_C</i>	<i>P_D</i>	<i>P_E</i>
13202.21	69364.61	54153.72	70813.64

Now, the actual distributed loads and their equivalent are applied to the FEM model, as shown in Figure 19, separately.

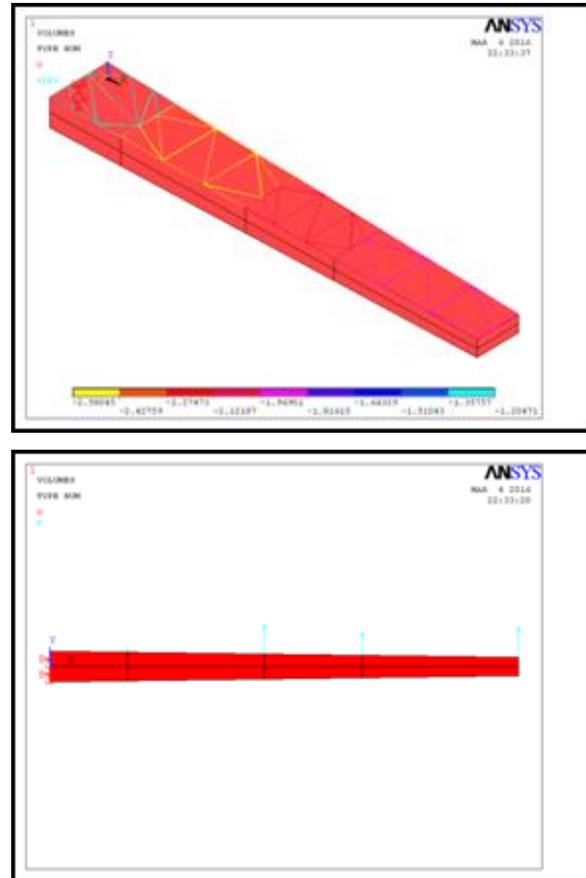


Fig. 19. Tapered cantilever beam model subjected to various distributed and equivalent point loads

Furthermore, the magnitudes of typical beam deflections for the two aforesaid loading conditions are achieved while the numerical solutions are accomplished. In Figures 20 and 21, the obtained deflections due to applying distributed and equivalent point loads and the corresponding beam deflection distributions are shown, respectively.

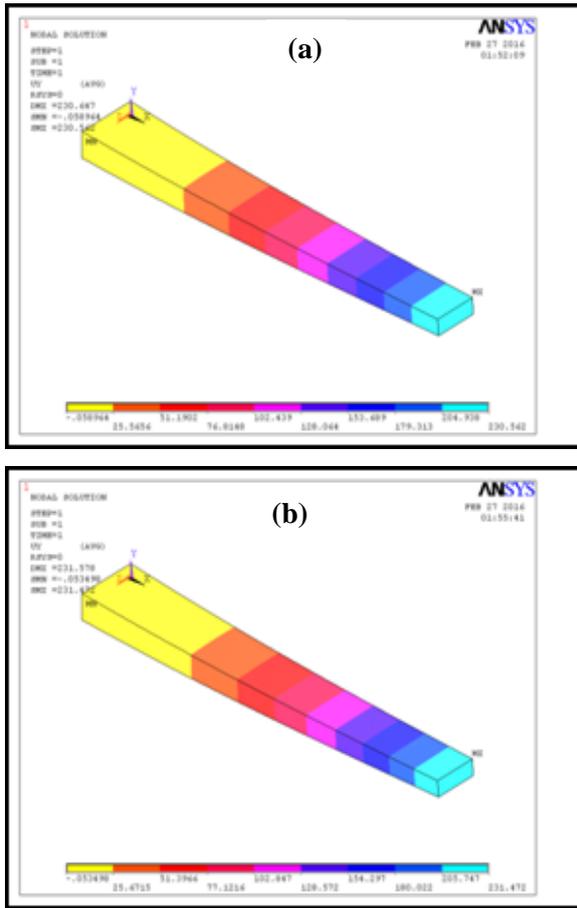


Fig. 20. Resulted deflections due to applying (a) distributed and (b) equivalent point loads

Example 4 denotes a tapered cantilever beam with constant optional properties with rectangular

cross sections and widths of 425 and 349 mm at the support and free end, respectively.

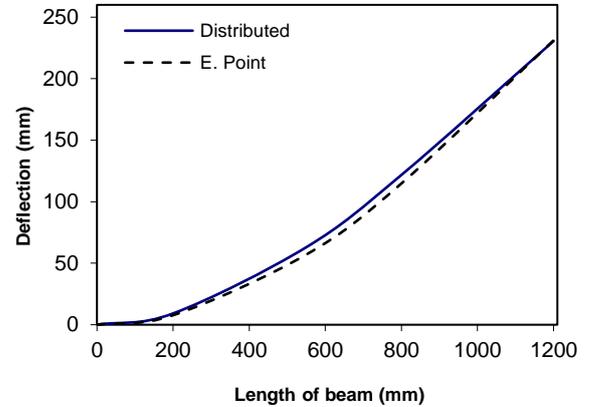


Fig. 21. Comparison of tapered beam deflection distribution by applying distributed and equivalent point loads

The depth at the support is 300 mm as well as 110 mm at the free end. The length of the beam is 1900 mm and is divided into 8 unequal parts, which are subjected to various distributed loads according to Table 6. As explained, the typical beam is assigned a width and depth greater than the previous example and with a greater rate of thickness changes.

The equivalent concentrated loads for all strips are achieved by employing Equation (35) and are filed in Table 6.

Table 6. Relative data for the beam in example#4

Strip No	W (N/mm)	q (N/mm ²)	L ₁ (mm)	L ₂ (mm)	L ₃ (mm)	P (N)
1	125	0.29682	0	200	1700	8326.623
2	200	0.482944	200	150	1500	19059.77
3	250	0.615582	350	250	1300	40373.54
4	150	0.380117	600	325	975	34031.64
5	225	0.584989	925	175	800	33721.92
6	175	0.467771	1100	350	450	48498.76
7	100	0.276158	1450	250	200	21827.7
8	275	0.778794	1700	200	0	50076.19

Now, the finite element model of the current beam is prepared, and the actual distributed and equivalent point loads are applied to the model separately. The

numerical solutions are carried out, and the beam deflection magnitudes and the corresponding beam deflection distributions are achieved.

The resulting deflections by applying the aforesaid loading conditions are depicted in Figures 22 and 23.

As can be observed in Figures 20 to 23, the agreements between the resulting deflection due to applying actual distributed and equivalent concentrated loads are quite satisfactory. Thus, the derived Equation in the current paper is approved as an appropriate approach for converting the distribution loads of cantilever beams and airplane wings to equivalent point loads.

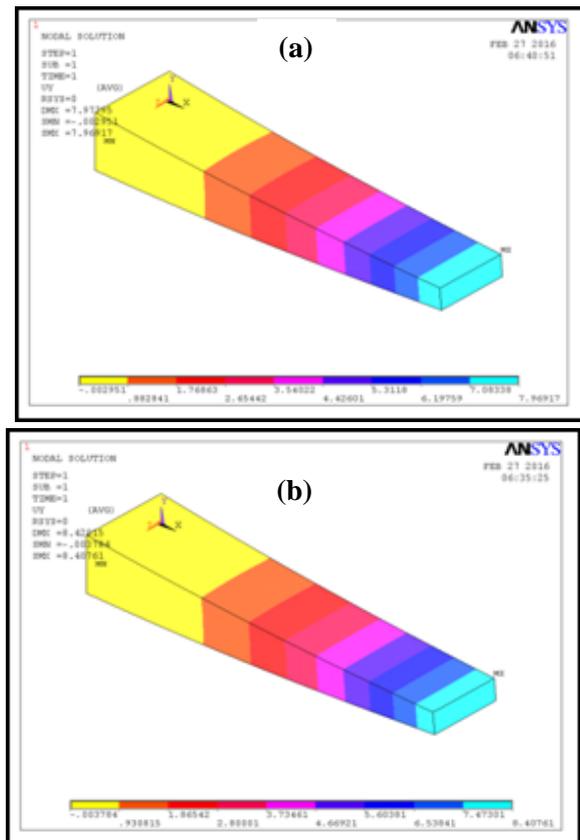


Fig. 22. Deduced deflections due to applying (a) distributed and (b) equivalent point loads to the sample beam No#4

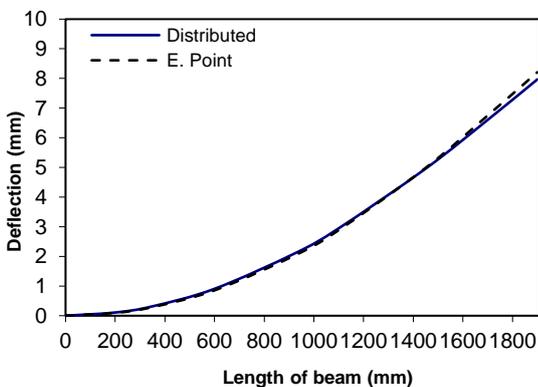


Fig. 23. Comparison of sample beam NO#4 deflection distribution by applying distributed and equivalent point loads

4 CONCLUSIONS

The development of equations via an analytical process for converting distributed load to concentrated load at the arbitrary acting point with equal resulting deflections has been attempted. These equations are designed for analytical calculations of cantilever simple and tapered beams and other structures with various distributed loads.

The validation study is carried out by arbitrarily selecting simple and tapered beams with scaled dimensions the same as aircraft wings and subjected to distributed loads. The mentioned loads are converted to concentrated loads at specific points via developed analytic equations. The actual distributed and corresponding equivalent point loads are applied to the finite element models, and the deflection distributions are compared. The agreement between the results seems to be quite satisfactory.

Thus, the developed equations are approved as being quite capable tools to convert distributed loads to point loads for all structures that are under various distributed loads on their different parts.

CONFLICT OF INTEREST

No potential conflict of interest was reported by the authors.

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