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A Simple Explicit Guidance Scheme Based on Velocities-to-be-Gained

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In this paper, a closed-loop strategy in the vertical plane is derived in order to determine the thrust direction of a launch vehicle in terms of velocities-to-be-gained. The two velocities-to-be-gained are utilized, here, for a given altitude and zero vertical speed in a specified final time. The formulation is obtained for constant gravity assumption, but it works when the velocities-to-be-gained are obtained for a spherical-Earth model via explicit or implicit relations.

INTRODUCTION

Guidance laws based on velocity-to-be-gained may be classified into two categories, namely explicit and implicit schemes. In explicit guidance, the required velocity is computed onboard explicitly whereas the velocity-to-be-gained is computed via a first-order differential equation in the well-known Q guidance, as an implicit one [1-4]. Implicit and explicit guidance algorithms have their advantages and disadvantages which are beyond the scope of this paper.

Another type of guidance law which determines a near optimal or an effective direction of the thrust vector for orbit injection [5-9] is mainly based on the linear-tangent law, or even linear law for thrust vector angle. In this class of guidance schemes, an explicit or iterative [10-12] algorithm is utilized for calculation of the thrust direction. A comparison of optimal solution with the results of two linear attitude programs for thrust angle (measured from the local horizon, and a fixed reference in an inertial reference) is given in Ref. [13].

The velocity-to-be-gained guidance technique presented in Ref. [2] is workable if it is possible to define, at each instant of thrusting, a required velocity to meet mission objectives which is only a function of current position, as stated by Battin. This requirement cannot be met for the problem having both final position and velocity constraints. In 1987, Bhat and Shrivastava developed a modified Q-guidance scheme to place a payload into a specified circular orbit [14]. By defining two velocities-to-be-gained and their corresponding first-order differential equations, the well-known Q-guidance scheme can be modified for elliptical orbits [15]. The implicit method of Ref. [15] needs a steering algorithm for launch vehicles when there is no control on the thrust magnitude. In other words, it needs an algorithm for determining the thrust vector direction in terms of the two velocities-to-be-gained, which is the subject of this research. For this purpose, the Cherry's E guidance method [16] is formulated in terms of velocities-to-be-gained, because it has more flexibility than other methods for further modifications .

The main objective of this research is to obtain a class of closed-loop guidance laws in terms of velocities-to-be-gained in order that the differential equations of velocities-to-be-gained can be applicable to the launch vehicle guidance algorithm. However, these velocities-to-be-gained may be preferred to be obtained from an explicit or iterative algorithm depending on applications.

BASIC FORMULATION

The governing equation of motion of a vehicle as a particle P in the vacuum is given by:

$$\ddot{\mathbf{r}} = \mathbf{g} + \mathbf{a}_T \tag{1}$$

where $\mathbf{r}$, $\mathbf{g}$, $\mathbf{a}_T$ are the vehicle position, gravitational acceleration, and acceleration due to thrust with respect to the flat-Earth model, respectively.

Consider the following change of variables:

$$ZEV_z = -v_z - g_z t_{go} \tag{2a}$$

$$ZEM_z = H - z - v_z t_{go} - \frac{1}{2} g_z t_{go}^2 \tag{2b}$$

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where z, v_z, and g_z are the components of the vehicle position $\mathbf{r}$, velocity $\mathbf{v}$, and gravitational acceleration in the vertical direction (*i.e.*, z axis) at the current time t (see Figure 1); H is the desired final altitude at the final time t_f; and t_{go} is the time-to-go until the final time ($t_{go} = t_f - t$).

Differentiation of Eqs. (2) with constant gravity assumption yields:

$$dZEV_z/dt = -a_{T_z} \tag{3a}$$

$$dZEM_z/dt = -t_{go}a_{T_z} \tag{3b}$$

where $a_{T_z} = a_T \sin\beta$ is the vertical component of the thrust acceleration, β is the thrust vector angle with respect to the horizon (x axis), and a_T is the acceleration magnitude due to thrust.

Equations (3) are integrated from the current time to the final time into:

$$ZEV_z(t_f) - ZEV_z(t) = -\int_t^{t_f} a_T(\xi)\sin\beta(\xi)d\xi \tag{4a}$$

$$ZEM_z(t_f) - ZEM_z(t) = -\int_t^{t_f}(t_f - \xi)a_T(\xi)\sin\beta(\xi)d\xi \tag{4b}$$

The final conditions $z(t_f) = H$ and $v_z(t_f) = 0$ are converted in terms of the new state variables into $ZEM_z(t_f) = 0$ and $ZEV_z(t_f) = 0$; therefore, applying the final conditions to Eqs. (4) rises to:

$$ZEV_z(t) = \int_t^{t_f} a_T(\xi)\sin\beta(\xi)d\xi \tag{5a}$$

$$ZEM_z(t) = \int_t^{t_f}(t_f - \xi)a_T(\xi)\sin\beta(\xi)d\xi \tag{5b}$$

If the acceleration due to thrust is assumed constant, we will have:

$$\int_t^{t_f} \sin\beta(\xi)d\xi = ZEV_z/a_T \tag{6a}$$

$$\int_t^{t_f}(t_f - \xi)\sin\beta(\xi)d\xi = ZEM_z/a_T \tag{6b}$$

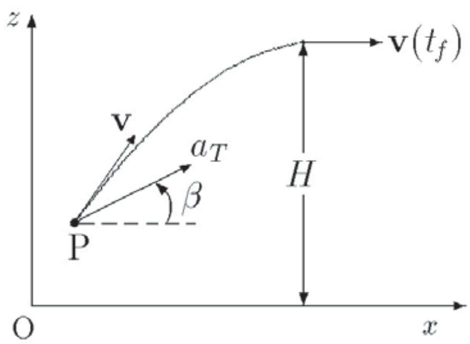

Figure 1. Problem geometry in a flat-Earth model.

When the thrust direction is given as a function of the time-to-go, it is better to rewrite Eqs. (6) in terms of the time-to-go instead of the current time, that is,

$$\int_0^{t_{go}} \sin\beta(\tau)d\tau = ZEV_z/a_T \tag{7a}$$

$$\int_0^{t_{go}} \tau\sin\beta(\tau)d\tau = ZEM_z/a_T \tag{7b}$$

where $\tau = t_f - \xi$.

CLOSED-LOOP STRATEGY

The thrust angle history is taken as a function of the time-to-go in the form of:

$$\sin\beta = C_0 f(t_{go}) + C_1 h(t_{go}) \tag{8}$$

provided that $|C_0 f(t_{go}) + C_1 h(t_{go})| \leq k < 1$ where $f(\cdot)$ and $h(\cdot)$ are linearly independent, pre-specified functions of the time-to-go, and k is determined by the guidance designer. Also, C_0 and C_1 are constants, obtained from Eqs. (7) to satisfy the final conditions, that is:

$$C_0\left[\int_0^{t_{go}} f(\tau)d\tau\right] + C_1\left[\int_0^{t_{go}} h(\tau)d\tau\right] = \frac{ZEV_z}{a_T} \tag{9a}$$

$$C_0\left[\int_0^{t_{go}} \tau f(\tau)d\tau\right] + C_1\left[\int_0^{t_{go}} \tau h(\tau)d\tau\right] = \frac{ZEM_z}{a_T} \tag{9b}$$

Solving for C_0 and C_1 yields

$$C_0 = \frac{I_{\tau h}(t_{go})ZEV_z - I_h(t_{go})ZEM_z}{a_T D(t_{go})} \tag{10}$$

$$C_1 = \frac{-I_{\tau f}(t_{go})ZEV_z + I_f(t_{go})ZEM_z}{a_T D(t_{go})} \tag{11}$$

where

$$I_f(t_{go}) = \int_0^{t_{go}} f(\tau)d\tau \tag{12a}$$

$$I_h(t_{go}) = \int_0^{t_{go}} h(\tau)d\tau \tag{12b}$$

$$I_{\tau f}(t_{go}) = \int_0^{t_{go}} \tau f(\tau)d\tau \tag{12c}$$

$$I_{\tau h}(t_{go}) = \int_0^{t_{go}} \tau h(\tau)d\tau \tag{12d}$$

$$D(t_{go}) = I_f(t_{go})I_{\tau h}(t_{go}) - I_{\tau f}(t_{go})I_h(t_{go}) \tag{12e}$$

Note that the functions $f(\cdot)$ and $h(\cdot)$ must be chosen in such a way that $D(t_{go}) \neq 0$ for $0 \leq t < t_f$.

Hence, substitution of Eqs. (10) and (11) into Eq. (8) results in a closed-loop relation for thrust angle as follows:

$$\sin\beta = \frac{I_{\tau h}(t_{go})f(t_{go}) - I_{\tau f}(t_{go})h(t_{go})}{a_T D(t_{go})} ZEV_z + \frac{I_f(t_{go})h(t_{go}) - I_h(t_{go})f(t_{go})}{a_T D(t_{go})} ZEM_z \qquad (13)$$

Also, the time history of the thrust angle is given by:

$$\sin\beta = \frac{I_{\tau h}(t_f)f(t_{go}) - I_{\tau f}(t_f)h(t_{go})}{a_T D(t_f)} ZEV_{z_0} + \frac{I_f(t_f)h(t_{go}) - I_h(t_f)f(t_{go})}{a_T D(t_f)} ZEM_{z_0} \qquad (14)$$

Since C_0 and C_1 are constants, we have:

$$C_0 = \frac{I_{\tau h}(t_{go})ZEV_z - I_h(t_{go})ZEM_z}{a_T D(t_{go})} = \frac{I_{\tau h}(t_f)ZEV_{z_0} - I_h(t_f)ZEM_{z_0}}{a_T D(t_f)} \qquad (15)$$

$$C_1 = \frac{-I_{\tau f}(t_{go})ZEV_z + I_f(t_{go})ZEM_z}{a_T D(t_{go})} = \frac{-I_{\tau f}(t_f)ZEV_{z_0} + I_f(t_f)ZEM_{z_0}}{a_T D(t_f)} \qquad (16)$$

Rearrangement of the two preceding relations in the matrix form yields:

$$\begin{bmatrix} I_{\tau h}(t_{go}) & -I_h(t_{go}) \\ -I_{\tau f}(t_{go}) & I_f(t_{go}) \end{bmatrix} \begin{bmatrix} ZEM_z \\ ZEV_z \end{bmatrix} = \frac{D(t_{go})}{D(t_f)} \begin{bmatrix} I_{\tau h}(t_f)ZEV_{z_0} - I_h(t_f)ZEM_{z_0} \\ -I_{\tau f}(t_f)ZEV_{z_0} + I_f(t_f)ZEM_{z_0} \end{bmatrix} \qquad (17)$$

Hence, the solutions for ZEM_z and ZEV_z result in:

$$ZEV_z = \frac{I_f(t_{go})I_{\tau h}(t_f) - I_h(t_{go})I_{\tau f}(t_f)}{D(t_f)} ZEV_{z_0} + \frac{I_h(t_{go})I_f(t_f) - I_f(t_{go})I_h(t_f)}{D(t_f)} ZEM_{z_0} \qquad (18)$$

$$ZEM_z = \frac{I_{\tau f}(t_{go})I_{\tau h}(t_f) - I_{\tau h}(t_{go})I_{\tau f}(t_f)}{D(t_f)} ZEV_{z_0} + \frac{I_{\tau h}(t_{go})I_f(t_f) - I_{\tau f}(t_{go})I_h(t_f)}{D(t_f)} ZEM_{z_0} \qquad (19)$$

From Eqs. (2) we have:

$$v_z = -ZEV_z - g_z t_{go} \qquad (20)$$

$$z = H - ZEM_z + ZEV_z t_{go} + \frac{1}{2} g_z t_{go}^2 \qquad (21)$$

Substitution of Eqs. (18) and (19) for ZEV_z and ZEM_z into the preceding relations, respectively, gives the time history of vertical position and speed.

SPECIAL CASES

Case a) $\sin\beta = C_0 + C_1 t_{go}$:

As a special case, a linear sinus law is chosen. The closed-loop guidance law (13) is, therefore, simplifies to:

$$\sin\beta = \frac{6ZEM_z - 2ZEV_z t_{go}}{a_T t_{go}^2} \qquad (22)$$

The time history of the thrust angle is found to be:

$$\sin\beta = \frac{-6 + 12(t_{go}/t_f)}{a_T t_f^2} ZEM_{z_0} + \frac{4 - 6(t_{go}/t_f)}{a_T t_f} ZEV_{z_0} \qquad (23)$$

Case b) $\sin\beta = C_0 + C_1 t_{go}^n$:

Here, a linear sinus law is modified in the form of $\sin\beta = C_0 + C_1 t_{go}^n$ where $n > 0$ is an additional constant for trajectory optimization. The guidance law is, then, given by:

$$\sin\beta = \frac{2(n+2)ZEM_z - (n+1)ZEV_z t_{go}}{a_T t_{go}^2} \qquad (24)$$

The time history of the thrust angle can be obtained as follows:

$$\sin\beta = \frac{2(n+2)[-1 + 2(n+1)(t_{go}/t_f)^n]}{n a_T t_f^2} ZEM_{z_0} + \frac{(n+1)[2 - (n+2)(t_{go}/t_f)^n]}{n a_T t_f} ZEV_{z_0} \qquad (25)$$

Case c) $\sin\beta = C_0 t_{go}^m + C_1 t_{go}^n$:

Consider the thrust angle to be in the form of $\sin\beta = C_0 t_{go}^m + C_1 t_{go}^n$ where two additional constants $m \geq 0$ and $n > 0$ are used here ($m \neq n$). Therefore, Cases a and b are considered as special cases of Case c. From Eqs. (12) we obtain:

$$I_f(t_{go}) = t_{go}^{m+1}/(m+1) \qquad (26a)$$

$$I_h(t_{go}) = t_{go}^{n+1}/(n+1) \qquad (26b)$$

$$I_{\tau f}(t_{go}) = t_{go}^{m+2}/(m+2) \qquad (26c)$$

$$I_{\tau h}(t_{go}) = t_{go}^{n+2}/(n+2) \qquad (26d)$$

$$D(t_{go}) = \frac{(n-m)t_{go}^{m+n+3}}{(m+1)(n+1)(m+2)(n+2)} \qquad (26e)$$

Hence, Eqs. (13) and (14) are simplified to the following relations, respectively:

$$\sin\beta = \frac{(m+2)(n+2)}{a_T t_{go}^2} ZEM_z - \frac{(m+1)(n+1)}{a_T t_{go}} ZEV_z \qquad (27)$$

$$\sin\beta = \frac{(m+1)[(m+2)(t_{go}/t_f)^m-(n+2)(t_{go}/t_f)^n]}{(n-m)a_T t_f/(n+1)}ZEV_{z_0} - \frac{(m+2)[(m+1)(t_{go}/t_f)^m-(n+1)(t_{go}/t_f)^n]}{(n-m)a_T t_f^2/(n+2)}ZEM_{z_0} \quad (28)$$

GUIDANCE LAW IN TERMS OF VELOCITIES-TO-BE-GAINED

The required velocity v_r^p is defined as an instantaneous velocity, required to satisfy the final position constraint H without any control effort in the vacuum. In a similar manner, the required velocity v_r^v is defined to satisfy the final velocity constraint without any control effort. For the flat-Earth model, these required velocities are obtained by setting Eqs. (2) equal to zero, that is,

$$-v_{r_z}^v - g_z t_{go} = 0 \quad (29a)$$

$$H - z - v_{r_z}^p t_{go} - \frac{1}{2}g_z t_{go}^2 = 0 \quad (29b)$$

Rearrangement yields:

$$v_{r_z}^v = -g_z t_{go} \quad (30a)$$

$$v_{r_z}^p = \frac{H-z}{t_{go}} - \frac{1}{2}g_z t_{go} \quad (30b)$$

Each velocity-to-be-gained is defined as a difference between the corresponding required velocity and the vehicle current velocity, that is,

$$v_{g_z}^v = v_{r_z}^v - v_z \quad (31a)$$

$$v_{g_z}^p = v_{r_z}^p - v_z \quad (31b)$$

By inspection, one can simply find that:

$$ZEV_z = v_{g_z}^v \quad (32a)$$

$$ZEM_z = v_{g_z}^p t_{go} \quad (32b)$$

Substitution of the preceding relations into Eq. (13) results in:

$$\sin\beta = \frac{I_{\tau h}(t_{go})f(t_{go}) - I_{\tau f}(t_{go})h(t_{go})}{a_T D(t_{go})}v_{g_z}^v + \frac{I_f(t_{go})h(t_{go}) - I_h(t_{go})f(t_{go})}{a_T D(t_{go})}v_{g_z}^p t_{go} \quad (33)$$

For instance, we have the following relation for Case c:

$$\sin\beta = \frac{(m+2)(n+2)}{a_T t_{go}}v_{g_z}^p - \frac{(m+1)(n+1)}{a_T t_{go}}v_{g_z}^v \quad (34)$$

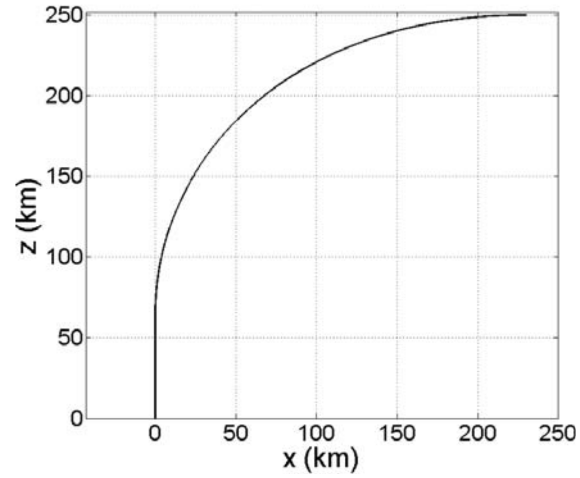

Figure 2. Trajectory of the launch vehicle for m=0 and n=2.

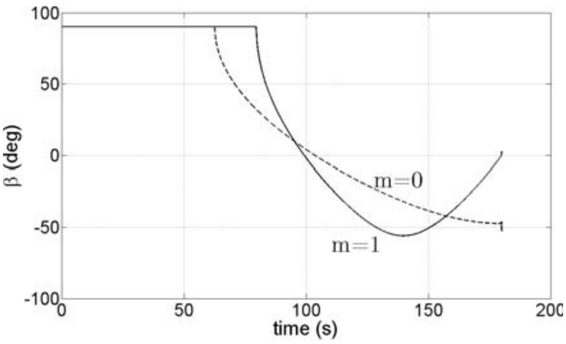

Figure 3. Thrust angle profiles for m=0,1 and n=2.

SIMULATION RESULTS

Consider a launch vehicle having an initial position $\mathbf{r} = \mathbf{0}$ and velocity components $v_x = 0$, $v_z = 1$ m/s lifts off the Earth surface. The acceleration due to the thrust is taken to be 40 m/s^2. The simulation results for the three guidance methods, *i.e.* Cases a, b, and c, are presented for the flat-Earth model with constant gravity assumption. These guidance laws are equivalent to Eq. (34), which is in terms of the velocities-to-be-gained. If the right-hand side of Eq. (34) exceeds the values of ± 1, it saturates at these values. In the simulation, the presented guidance scheme is applied from the lift off to show the effect of the saturation. However, a pitch programming algorithm is utilized, in practice, in the first stage of flight.

We are to reach a final altitude of 250 km without any vertical speed. The simulation results are listed in Table 1 for the specified final time of 180 s. For $m = 0$ and $n = 1$, there is a considerable final error in the vertical speed, *i.e.* 4.3 m/s. However, the error in altitude, e_H, is negligible. By selecting proper values for m and n, the final constraints can be achieved provided that the final time is sufficient. For example, using $m = 0$ and $n = 2$ the final constraints are met satisfactorily. The launch vehicle trajectory is depicted in Figure 2. The behavior of the thrust angle is also shown in Figure 3. To zero out the thrust vector angle, the parameter m is taken a positive value. However, because of the saturation or singularity of the guidance gains at the final instants, the thrust vector angle may have not a zero value. To remove the singularity, a constant thrust angle may be applied in the final instants. However, as stated by Cherry [16], since a very precise control of altitude is normally not required, it may be desirable to abandon altitude control when the time-to-go becomes smaller than some preset value resulting in a more accurate control of the vertical speed.

In order to reach a desired horizontal speed for orbit injection, the thrusting may continue in a way that the vertical thrust acceleration cancels out the gravitational acceleration. However, the final time t_f is usually chosen to control the horizontal speed.

The parameters m and n can be utilized for trajectory optimization as well as the final time. The effect of the final time on the horizontal speed can be viewed in Table 2 ($e_H = 0.00$ m & $v_z = 0.00$ m/s). Increasing the final time increases the final horizontal speed. It is known that an increase of Δt to the final time can, at most, increase the horizontal speed by $\Delta V = a_T \Delta t$ (however, this is true if the previous β profile does not vary with the addition of the extra Δt). As seen in Table 2, an increase of 20 s to $t_f = 180$ s increases the horizontal speed by 1757 m/s whereas $\Delta V = 800$ m/s. It means that the thrust direction programming is more efficient for $t_f = 200$ s (or 250 s) than that for $t_f = 180$ s. However, increasing 10 s to $t_f = 290$ s increases the horizontal speed by 435 m/s whereas $\Delta V = 400$ m/s.

The values of the minimum time T^*, obtained from the minimum-time open-loop solution [17], are presented for the corresponding values of the horizontal speed v_x, listed in the second column of Table 2. For example, the minimum time to attain a horizontal speed of 3430.3 m/s at an altitude of 250 km is 171.55 s. The minimum time solution is obtained for $\mathbf{v}_0 = \mathbf{0}$; however, the initial vertical speed for the non-optimum solutions is set to be 1 m/s. As it can be seen in Table 2, there is little difference between the optimal minimum time and the final time of the non-optimum solution. The performance may be enhanced if the maximum value of β is restricted to a predetermined value of β_{max}. Table 3 presents the horizontal speed for various values of $k = \sin\beta_{\max}$ at $m = 0.2$, $n = 1$, and $t_f = 180$ s. Also, the corresponding minimum time values are given in this table. For $k = 0.93$, the difference between

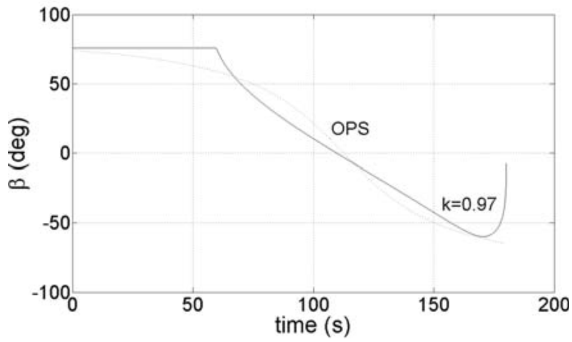

Figure 4. Thrust angle profiles for $k = 0.97$ and open-loop time-optimal solution (OPS).

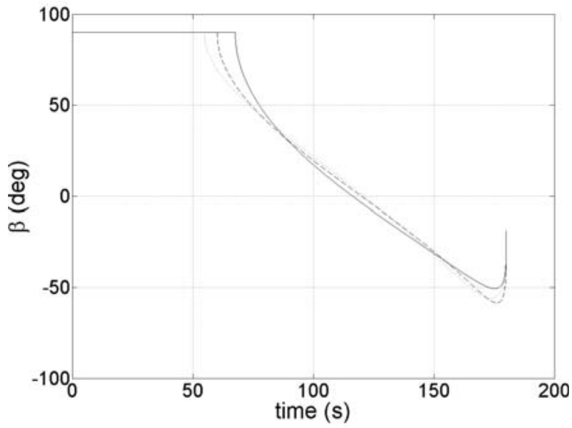

Figure 5. The thrust angle history for three guidance equations; solid line: Eq. (27), dashed line: Eq. (37), dotted line: modified Eq. (27).

Table 1. Simulation results for guidance equation (27) with $t_f = 180$ s.

Guidance Parameters	v_x (m/s)	v_z (m/s)	e_H (m)	$\beta(t_f)$ (deg)
$m = 0, n = 1$	4032.2	4.31	-5.31	-90
$m = 0, n = 2$	3879.6	0.01	0.00	-53
$m = 1, n = 2$	3157.6	0.00	0.00	2.6
$m = 0, n = 1.15$	4043.0	0.02	0.00	-90
$m = 0.2, n = 1$	4013.0	0.00	0.00	-7.1

the minimum-time optimum and non-optimum values is 0.64 s, which shows enhancement in the horizontal speed. The thrust angle profiles for both methods are depicted in Figure 4 for $k = 0.97$.

In the next step, the thrust acceleration profile is assumed to vary from an initial value of 30 m/s^2 to 60 m/s^2 during the time 0 to 180 s. Equation (27) may be modified by replacing $a_T(t)$ with the average acceleration $\bar{a}_T(t) = [a_T(t) + a_T(t_f)]/2$ ($t_{go}\bar{a}_T(t) = \int_t^{t_f} a_T(t')dt'$). The horizontal speed is listed in Table 4 for three guidance equations, *i.e.* Eqs. (27), (37), and the modified Eq. (27) with $k = 1$. The results are case dependent. The performance of Eq. (27) is degraded with respect to the modified Eq. (27) or Eq. (37). The thrust angle profiles for the three-mentioned equations are depicted in Figure 5.

Subsequently, a non-rotating spherical-Earth model is considered in the flight simulation code. The initial and final values are the same as the values for the flat-Earth model, but are described here in local coordinates. Equations (30) are also computed in local coordinates. The values of the local horizontal speed, v_{x_L}, are listed in Table 5 for the three-mentioned guidance equations. Similarly, the results are case dependent; however, the performance can be enhanced using suitable values of k, m, and n. The trajectory of the launch vehicle for the modified Eq. (27) with $m = 0.1$ and $n = 0.9$ is illustrated in Figure 6 in which z_L is the local altitude and S is the range angle multiplied by the Earth radius. Also, the thrust angle profiles are compared in Figure 7 for the three-mentioned guidance equations with $m = 0.1$, $n = 0.9$, and $t_f = 180$ s. Here, the thrust angle β_L has been taken with respect to the local horizon; however, it may considered with respect to a fixed reference in inertial space.

As a future study, the functions $f(\cdot)$ and $h(\cdot)$

Table 2. Simulation results for $m = 0.2$ and $n = 1$ with different final times.

t_f (s)	v_x (m/s)	$\beta(t_f)$ (deg)	T^* (s)
175	3430.3	−8.12	171.55
180	4013.0	−7.12	175.57
200	5770.0	−4.3	193.90
250	8948.9	−1.7	249.11
290	10798	−0.71	289.75
300	11233	−0.52	299.82

Table 3. The horizontal speed for various values of $k = \sin\beta_{\max}$ ($m = 0.2$, $n = 1$, $t_f = 180$)

k	v_x (m/s)	T^* (s)
0.99	4266.7	177.63
0.97	4392.3	178.72
0.95	4446.8	179.21
0.93	4464.5	179.37
0.91	4448.1	179.22

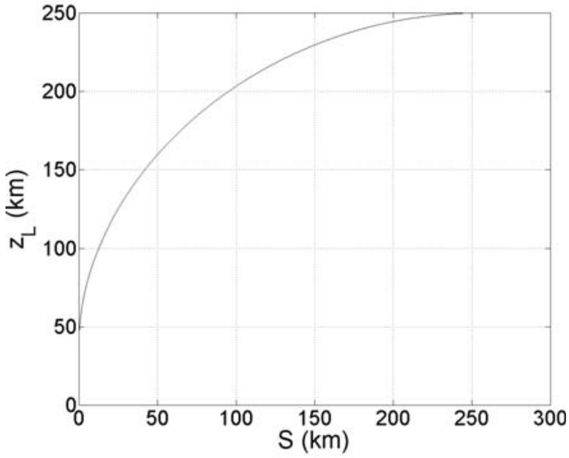

Figure 6. Trajectory of the launch vehicle in spherical Earth model using Modified Eq. (27) with m=0.1, n=0.9.

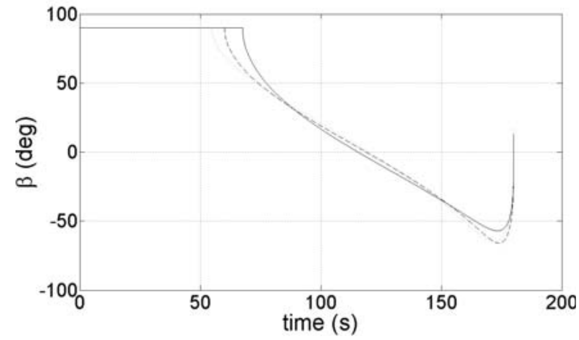

Figure 7. The thrust angle history for three guidance equations using spherical Earth model; solid line: Eq. (27), dashed line: Eq. (37), dotted line: the modified Eq. (27).

Table 4. Comparison of three guidance equations for $t_f = 180$ s.

Guidance Eq.	Guidance Parameters	v_x (m/s)
Eq. (27)	$m = 0, n = 1$	4734
	$m = 0, n = 0.9$	4752
	$m = 0.1, n = 0.9$	4720
Mod Eq. (27)	$m = 0, n = 1$	4900
	$m = 0, n = 0.9$	4914
	$m = 0.1, n = 0.9$	4888
Eq. (37)	$m = 0, n = 1$	4817
	$m = 0.1, n = 0.9$	4824
	$m = 0.3, n = 0.5$	4857

or the parameters m and n may be determined from the optimal numerical solutions. In addition, the parameters k, m, and n are proposed to be chosen as variables with time and velocities-to-be-gained.

CONCLUSIONS

This work introduces a closed-loop guidance strategy for a launch vehicle in terms of velocities-to-be-gained. This approach is a basis for guidance designers to develope a class of guidance laws in terms of velocities-to-be-gained for nonthrottleable rockets. Moreover, the analytical solution of position and velocity components in the vertical direction are obtained. The solution can be extended for time-varying thrust magnitude. In addition, an implicit guidance with the two differential equations of velocities-to-be-gained with final position and velocity constraints may be implemented using the present closed-loop strategy of the thrust direction.

APPENDIX: TIME-VARYING THRUST

Explicit solutions can be simply obtained for time-varying thrust magnitude for constant gravity model. For example, consider a linear profile for the acceleration due to thrust, that is:

$$a_T(t) = a_0 - b_0 t_{go} \tag{35}$$

where $a_0 = a_T(t_f)$ and b_0 are constant. For simplicity, the solution for Case c is, here, presented, *i.e.* $\sin\beta = C_0 t_{go}^m + C_1 t_{go}^n$. Substitution into Eqs. (5a) gives:

$$\int_0^{t_{go}} (a_0 - b_0\tau)(C_0\tau^m + C_1\tau^n)d\tau = ZEV_z \tag{36a}$$

$$\int_0^{t_{go}} (a_0 - b_0\tau)(C_0\tau^{m+1} + C_1\tau^{n+1})d\tau = ZEM_z \tag{36b}$$

The closed-loop solution is found to be:

$$\sin\beta = \frac{N_p ZEM_z + N_v ZEV_z t_{go}}{D_v t_{go}^2} \tag{37}$$

Table 5. Comparison of three guidance equations using spherical Earth model ($t_f = 180$ s).

Guidance Eq.	Guidance Parameters	v_x (m/s)
Eq. (27)	$m = 0, n = 1$	4562
	$m = 0.1, n = 0.9$	4547
	$m = 0.1, n = 0.6$	4578
Mod Eq. (27)	$m = 0, n = 1$	4683
	$m = 0, n = 0.9$	4677
	$m = 0.1, n = 0.9$	4671
Eq. (37)	$m = 0, n = 1$	4617
	$m = 0.1, n = 0.9$	4620
	$m = 0.2, n = 0.7$	4622

where

$$N_p = (n+2)(m+2)a_0 - (n+1)(m+1)b_0 t_{go} \tag{38}$$

$$N_v = -(m+1)(n+1)a_0 + \frac{(n+1)(m+1)(n+2)(m+2)}{(n+3)(m+3)} b_0 t_{go} \tag{39}$$

$$D_v = a_0^2 - \frac{2(n+2)(m+2)}{(n+3)(m+3)} a_0 b_0 t_{go} + \frac{(n+1)(m+1)}{(n+3)(m+3)} b_0^2 t_{go}^2 \tag{40}$$

provided that $D_v \neq 0$ for $t_0 \leq t < t_f$. For $m = 0$ and $n = 1$, the preceding relation is simplified to:

$$\sin\beta = \frac{(-2a_0 + b_0 t_{go})t_{go} ZEV_z + (6a_0 - 2b_0 t_{go})ZEM_z}{t_{go}^2(a_0^2 - a_0 b_0 t_{go} - \frac{1}{6} b_0^2 t_{go}^2)} \tag{41}$$

The term $b_0 t_{go}$ may be replaced with $a_T(t_f) - a_T(t)$ in the closed-loop representation.

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