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Research Note

Buckling and Vibration Analyses of Angle-Ply Symmetric Laminated Composite Plates with Fully Elastic Boundaries

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The main focus of this paper is on efficiency analysis of two kinds of approximating functions (characteristic orthogonal polynomials and characteristic beam functions) that have been applied in the Rayleigh-Ritz method to determine the non-dimensional buckling and frequency parameters of an angle ply symmetric laminated composite plate with fully elastic boundaries. It has been observed that orthogonal polynomials yield superior results for the lower modes. Also, the overall CPU time consumed to perform the calculations by the two different procedures for constructing the approximating functions showed that orthogonal polynomials are computationally more time efficient. A novel approach is devised for the construction of characteristic beam functions for buckling and vibration analysis of an angle ply symmetric laminated composite plate. Numerical results are presented and discussed.

INTRODUCTION

The main purpose of this article is to discuss some aspects of the interesting paper developed by Bhat [1]. Bhat developed an interesting approach in vibration analysis of a plate by using orthogonal polynomials as approximating functions in the Rayleigh-Ritz (R-R) method. He showed that the results yielded by orthogonal polynomials are superior to characteristic beam functions in the R-R method. However, his conclusion was made for a plate made up of conventional materials and with simple boundary conditions. In this paper, we have generalized his conclusion to buckling and vibration of an angle ply symmetric laminated composite plate with fully elastic boundaries which is much more general than the case considered by Bhat. The model that has been studied in this article is shown in Figure 1 where an angle ply symmetric laminated composite plate is restrained by rotational and translational springs on the edges. The material of the plate is fiber reinforced composite. The laminate is of uniform thickness h and, in general is made up of a number of layers each consisting of unidirectional fiber reinforced composite material. The fiber angle of the kth layer is θ measured from the x axis to the fiber orientation, with all lamina having equal thicknesses, (see Figure 1). The material properties with 1 as index are those calculated in the fiber orientation and those with 2 as index have been calculated in the direction perpendicular to the fiber orientation. In this paper, two different procedures to construct the approximating functions have been applied to determine the non-dimensional critical buckling load and frequency parameter of the model described earlier, in the R-R method.

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CONSTITUTIVE EQUATIONS

The constitutive equations for the composite plate are expressed in matrix form as:

$$\left\{ \begin{array}{c} N \\ M \end{array} \right\} = \left[\begin{array}{cc} A & B \\ B & D \end{array} \right] \left\{ \begin{array}{c} \epsilon \\ \kappa \end{array} \right\} \tag{1}$$

where A, B, and D are extensional, coupling, and bending rigidities that are defined as follows:

$$A_{ij} = \sum_{k=1}^{N_P} \bar{Q}_{ij}^k (z_k - z_{k-1}) \tag{2}$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{N_P} \bar{Q}_{ij}^k (z_k^2 - z_{k-1}^2) \tag{3}$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N_P} \bar{Q}_{ij}^k (z_k^3 - z_{k-1}^3) \tag{4}$$

where $\overline{Q}_{ij}^k$ stands for elements of transformed reduced stiffness matrix of the kth layer. The midplane strains and curvatures are related to the deflections and transverse shear deformations through the kinematic relations:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \end{Bmatrix} \tag{5}$$

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} - \frac{1}{S_x}\frac{\partial Q_x}{\partial x} \\ \frac{\partial^2 w}{\partial y^2} - \frac{1}{S_y}\frac{\partial Q_y}{\partial y} \\ 2(\frac{\partial^2 w}{\partial x \partial y}) - \frac{1}{S_x}\frac{\partial Q_x}{\partial y} - \frac{1}{S_y}\frac{\partial Q_y}{\partial x} \end{Bmatrix} \tag{6}$$

where w, Q_x and Q_y are respectively transverse deflection and transverse shear forces. S_x and S_y are transverse shear stiffnesses of the plate in x-z and y-z planes respectively. The plate that has been considered in the current paper is thin so we may neglect the in-plane deflections: u and v and so Q_x and Q_y. Also, the transverse deflection is assumed to remain constant through the thickness. To relate edge moments and forces to transverse deflection, we have used the boundary conditions provided in [2] as:

$$at(x=0)\begin{cases} K_{r1}\frac{\partial w}{\partial x} + \left\{D_{11}\frac{\partial^2 w}{\partial x^2}\right\} = 0 \\ K_{t1}\, w \; - \; \left\{D_{11}\frac{\partial^3 w}{\partial x^3}\right\} \; = \; 0 \end{cases} \tag{7}$$

$$at(x=a)\begin{cases} K_{r2}\frac{\partial w}{\partial x} + \left\{D_{11}\frac{\partial^2 w}{\partial x^2}\right\} = 0 \\ K_{t2}\, w \; - \left\{D_{11}\frac{\partial^3 w}{\partial x^3}\right\} \; = \; 0 \end{cases} \tag{8}$$

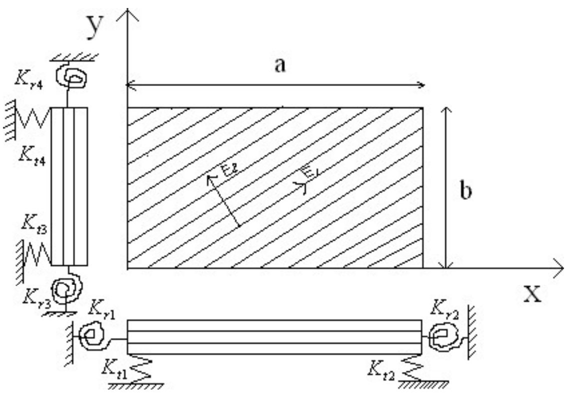

Figure 1. The model of the composite plate considered in this article.

After some manipulations the non-dimensional rigidities, R and T, will be appeared in distinguished terms that enables us to generate a computer code with more convenient handling on defining these terms as input data. R and T are defined as $R = \frac{K_r a}{D_{11}}$ and $T = \frac{K_t a^3}{D_{11}}$. Considering that:

$$R_1 = \frac{K_{r1}\, a}{D_{11}}, \quad R_2 = \frac{K_{r_2} a}{D_{11}}, \quad R_3 = \frac{K_{r_3} b}{D_{22}}, \quad R_4 = \frac{K_{r_4} b}{D_{22}}$$

$$T_1 = \frac{K_{t1}\, a^3}{D_{11}}, \quad T_2 = \frac{K_{t_2} a^3}{D_{11}}, \quad T_3 = \frac{K_{t_3} b^3}{D_{22}}, \quad T_4 = \frac{K_{t_4} b^3}{D_{22}}$$

and

$$R_1 = R_2 = R_3 = R_4 = R$$

$$T_1 = T_2 = T_3 = T_4 = T \tag{9}$$

CONSTRUCTION OF THE APPROXIMATING FUNCTIONS

characteristic beam orthogonal polynomials

In this paper, two different procedures have been applied to construct the approximating functions. One of them was first devised by Bhat [1] and the second has had a long history of application in vibration problems (characteristic beam function). Bhat proposed a set of orthogonal polynomials that was generated by using a Gram-Schmidt process to determine the frequency parameter of a plate made up of conventional materials and with simple boundary conditions in the R-R method. A procedure similar to what Bhat proposed in [1] is applied here but with small deviations. In the work of Bhat, the first member of the orthogonal polynomials set satisfied both geometrical and natural boundary conditions where the other members satisfied just the geometrical boundary conditions but in the current work the first member of the orthogonal polynomials set only satisfies the four natural boundary conditions of the corresponding beam problem which are as Eqs. (7 and 8). The procedure considered in the current paper is as follows:

First, we assume a polynomial with five terms as:

$$X_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \tag{10}$$

This polynomial should satisfy the four boundary conditions presented earlier in Eqs. (7 and 8). By replacing it in the four boundary conditions and determining the coefficients in terms of one undetermined coefficient as a_0 it is reduced to a more simple form as:

$$X_0(x) = a_0 f(x) \tag{11}$$

which after normalization has no constant and is the first member of the orthogonal polynomials set which can be generated by a Gram-Schmidt formula as follows:

$$X_1(x) = (x - B_1) X_0(x) \tag{12}$$

$$X_k(x) = (x - B_k)X_{k-1}(x) - C_k X_{k-2}(x) \tag{13}$$

$$B_k = \frac{\int_0^1 xW(x)X_{k-1}^2(x)dx}{\int_0^1 W(x)X_{k-1}^2(x)dx} \tag{14}$$

$$C_k = \frac{\int_0^1 xW(x)X_{k-1}(x)X_{k-2}(x)dx}{\int_0^1 W(x)X_{k-2}^2(x)dx} \tag{15}$$

where $W(x)$ is the weighting function that is set to unity due to uniform thickness of the composite plate considered in this paper. For plates of variable thickness, the weight function should be defined properly [3]. All the members of the orthogonal polynomials set satisfy the orthogonality condition as follows:

$$\int_0^1 W(x)X_k(x)X_l(x)dx = \left\{ \begin{array}{lll} 0 & if & k \neq l \\ a_{k1} & if & k = l \end{array} \right\} \tag{16}$$

The coefficients of the orthogonal polynomials set are chosen in such a way to make the polynomials orthonormalized as shown in the following equation:

$$\int_0^1 X_k^2(x)dx = 1 \tag{17}$$

The same procedure can be applied for the Y coordinate.

characteristic beam functions

To construct the characteristic beam functions which are suitable to be applied in the R-R method for buckling and vibration analysis of the corresponding plate, a general form of beam eigenfunction is firstly supposed as the first member of the approximating functions set as follows:

$$X_0(x) = a\sinh(\alpha_m x) + b\cosh(\alpha_m x) + c\cos(\alpha_m x) + d\sin(\alpha_m x) \tag{18}$$

Replacing this function in the four boundary conditions presented earlier, results in an algebraic system of equations with five unknowns. The determinant of the coefficient matrix that here is an expression that contains only α_m (one of the unknowns) is zero that leads us to construct the characteristic equation and finally determine the remaining unknowns $[a\,b\,c\,d]^{\,t}$ for each eigenmode. It should be noted that each eigenmode has a special eigenfunction. The novelty of this procedure would be confirmed when compared with the complex formulas devised in the existing literature.

APPLICATION OF THE RAYLEIGH-RITZ METHOD

In the application of the R-R method, first, the transverse deflection is represented by a frequency dependent co-ordinate function as:

$$w(x,y) = \sum_{i=1}^{M}\sum_{j=1}^{N} C_{ij}X_i(x)Y_j(y) \tag{19}$$

where C_{ij} is the generalized coordinate. The maximum kinetic energy of the freely vibrating plate with amplitude $w(x,y)$ and radian frequency ω, expressed in rectangular co-ordinates, is given by:

$$T_{\max} = \frac{\rho\, h\, \omega^2}{2} \iint_{R^*} w^2 dxdy \tag{20}$$

where ρ is the mass density of the plate material and the integration is carried out over the entire plate domain R^*. The maximum strain energy of the mechanical system under study is given by:

$$U_{\max} = U_{P,\,\max} + U_{R,\,\max} + U_{T,\,\max} \tag{21}$$

where $U_{p,\max}$ is the maximum strain energy due to the plate bending and also $U_{R,\,\max}$ and $U_{T,\,\max}$ are respectively the maximum strain energy stored in rotational and translational springs at the plate edges and have been given in [7].

In buckling case, the potential energy associated with the external loading is given by:

$$U_f = \int_0^a \int_0^b (\bar{N}_x w_{,x}^2 + \bar{N}_y w_{,y}^2 + 2\bar{N}_{xy} w_{,x} w_{,y})dxdy \tag{22}$$

The total energy functional for free vibration of the plate is given by:

$$F = U_{\max} - T_{\max} \tag{23}$$

which is to be minimized according to the R-R method. The transverse deflection of the plate is expressed by a set of characteristic orthogonal polynomials or characteristic beam functions as presented in Eq. 19. The minimization of the energy functional requires:

$$\frac{\partial}{\partial c_{ij}}(U_{\max} - T_{\max}) = 0,\ i,j = 1\ldots M,N. \tag{24}$$

which leads to the governing eigenvalue equation as followings:

$$(\,[K_b + K_{trns} + K_{rot}] - \lambda\ [M]\,)\ \{C_{i\,j}\} = 0 \tag{25}$$

Table 1. The first five frequency parameters of a composite plate (0/90/0) evaluated by using orthogonal polynomials in the Rayleigh Ritz method for various boundary conditions. Material properties are (G12 / E22 = 0.6, ν_{12} = 0.25). The values in bracket show the corresponding frequencies evaluated by using beam functions.

E11/E22	$\lambda 1$	$\lambda 1^{[5]}$	$\lambda 2$	$\lambda 3$	$\lambda 4$	$\lambda 5$
$R = 0$ $T = \infty$=(SIMPLY SUPPORTED)						
10	10.6498	10.6500	18.6409	34.2544	36.9938	42.5993
20	13.9482	13.9500	21.7556	38.6389	51.2066	55.7929
30	16.6046	16.6100	24.4832	42.5909	62.2556	70.2577
50	20.9296	-	29.1865	49.5667	81.4355	83.7187
100	29.0502	-	38.5112	63.7577	104.2774	112.4790
$R = 50$, $T = \infty$						
10	21.2430		30.2536	47.8961	54.1982	73.0140
20	27.5300		36.2877	54.9227	77.7562	82.4962
30	32.1890		41.1160	60.9090	85.6102	90.8191
50	39.4027		48.9562	71.0377	105.9967	111.0131
100	52.5191		63.7461	90.7381	133.5676	142.8313
$R = 0$, $T = 0.3$						
10	1.0823		1.5447	1.5459	5.4630	7.6646
20	1.0853		1.5458	1.5464	5.5046	8.6132
30	1.0872		1.5464	1.5466	5.5310	9.4674
50	1.0893		1.5470	1.5472	5.5654	10.9784
100	1.0917		1.5476	1.5480	5.6088	14.0630
$R = 40$, $T = 40$						
10	10.1927		14.4093	16.6413	19.9839	20.7500
			[14.5431]	[16.8516]	[20.0634]	[21.3282]
20	10.6535		14.8697	18.7049	21.8245	21.9081
30	10.9219		15.1672	20.2987	22.9562	23.2781
50	11.2639		15.5878	22.6719	24.8794	25.4908
100	11.9249		16.6207	29.8568	30.8373	33.2558

Table 2. The first five non-dimensional frequency parameters of a composite plate (30/-30/30) evaluated in the Ritz method by using beam eigenfunctios. Material properties are (G12 / E22 = 0.6, ν_{12} = 0.25), $R = T = 40$.

E11/E22	$\lambda 1$	$\lambda 2$	$\lambda 3$	$\lambda 4$	$\lambda 5$
10	10.1068	14.1895	16.1023	19.5830	22.5617
20	10.5994	14.6444	17.3061	20.8930	24.4090
30	10.8759	14.9426	18.2153	21.9183	26.0417
50	11.2157	15.3502	19.6214	23.6164	28.8045
100	11.6493	15.9688	21.9731	26.9331	33.9034

and K_b, K_{trns}, K_{rot} and M are defined as below:

$$[K_b]_{mnij} = \frac{1}{E_2 h^3} \{D_{11} \left(E_{mi}^{22} F_{nj}^{00}\right) + \alpha^2 D_{12} \left(E_{mi}^{02} F_{nj}^{20} + E_{mi}^{20} F_{nj}^{02}\right) + \alpha^4 D_{22} E_{mi}^{00} F_{nj}^{22} + 2\alpha D_{16} \left(E_{mi}^{21} F_{nj}^{01} + E_{mi}^{12} F_{nj}^{10}\right) + 2\alpha^3 D_{26} (E_{mi}^{01} F_{nj}^{21} + E_{mi}^{10} F_{nj}^{12}) + 4\alpha^2 D_{66} E_{mi}^{11} F_{nj}^{11}) \} \quad (26)$$

$$[K_{trns}]_{mnij} = T F_{nj}^{00} (X_m(0) Y_i(0) + X_m(1) Y_i(1)) + \alpha^4 T E_{mi}^{00} (Y_n(0) Y_j(0) + Y_n(1) Y_j(1)) \quad (27)$$

$$[K_{rot}]_{mnij} = R F_{nj}^{00} \left(\frac{\partial X_m}{\partial x} |_{x=0} \frac{\partial X_i}{\partial x} |_{x=0} + \frac{\partial X_m}{\partial x} |_{x=1} \frac{\partial X_i}{\partial x} |_{x=1} \right) + \alpha^4 R E_{mi}^{00} \left(\frac{\partial Y_n}{\partial y} |_{y=0} \frac{\partial Y_j}{\partial y} |_{y=0} + \frac{\partial Y_n}{\partial y} |_{y=1} \frac{\partial Y_j}{\partial y} |_{y=1} \right) \quad (28)$$

$$[M_{mnij}] = E_{mi}^{00} F_{nj}^{00} \quad (29)$$

And for biaxial buckling load:

$$[M]_{mnij} = E_{mi}^{11} F_{nj}^{00} + E_{mi}^{00} F_{nj}^{11} \quad (30)$$

It is emphasized that for uniaxial compression only the first term is applied. The unfamiliar terms in these expressions can be evaluated by the following formulas:

$$E_{m\,i}^{r\,s} = \int_0^1 \left[\frac{d^r X_i(x)}{dx^r} \frac{d^s X_j(x)}{dx^s} \right] dx \tag{31}$$

$$F_{n\,j}^{r\,s} = \int_0^1 \left[\frac{d^r Y_i(y)}{dy^r} \frac{d^s Y_j(y)}{dy^s} \right] dy \tag{32}$$

where $r, s = 0, 1, 2$. The non-dimensional frequency parameter and critical buckling load are respectively as follows:

$\sqrt{\lambda} = \varpi\, a^2 \sqrt{\frac{\rho}{E_2 h^2}}$

$\mu = \frac{N_{cr} a^2}{E_{22} h^3}$

NUMERICAL RESULTS

The comparison of the numerical results, achieved in this analysis and presented in Table 1, shows the superiority of the orthogonal polynomials over beam eigenfunctions as they present the lower quantities for the lower modes. Table 2 presents the numerical results of vibration analysis of an angle ply symmetric laminated composite plate with fully elastic boundaries. Figure 2 depicts the mode shapes of a composite plate with classical boundary conditions. Figures 3 and 4 show the convergence of non-dimensional critical buckling load for a certain composite plate with all edges clamped. It's worthy of note that the same behavior has been observed by Darvizeh *et.al.* [4] and also it should be noted that as, Figures 3 and 4 imply, the analysis performed in this paper can be generalized to single-layered plates which exhibit more general characteristics. Table 1 shows the frequency parameter for various non-dimensional elastic rigidities. In all buckling cases, material properties are as E11= 130 GPa, E22=9 GPa, G12=4.8 GPa, ν_{12} =0.28. Inconsistencies of the results in reference [7] can be revealed with a more careful look at Table 1.

CONCLUSIONS

In this paper, a new procedure has been developed for the construction of the approximating functions that were first applied to a plate made up of conventional materials and with simple boundary conditions by Bhat [1]. Although the same procedure has been

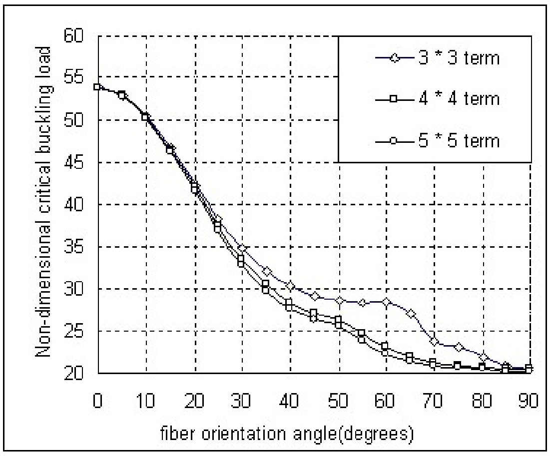

Figure 3. Convergence of non-dimensional critical Buckling load for a single layered composite plate with all edges clamped, (Beam functions).

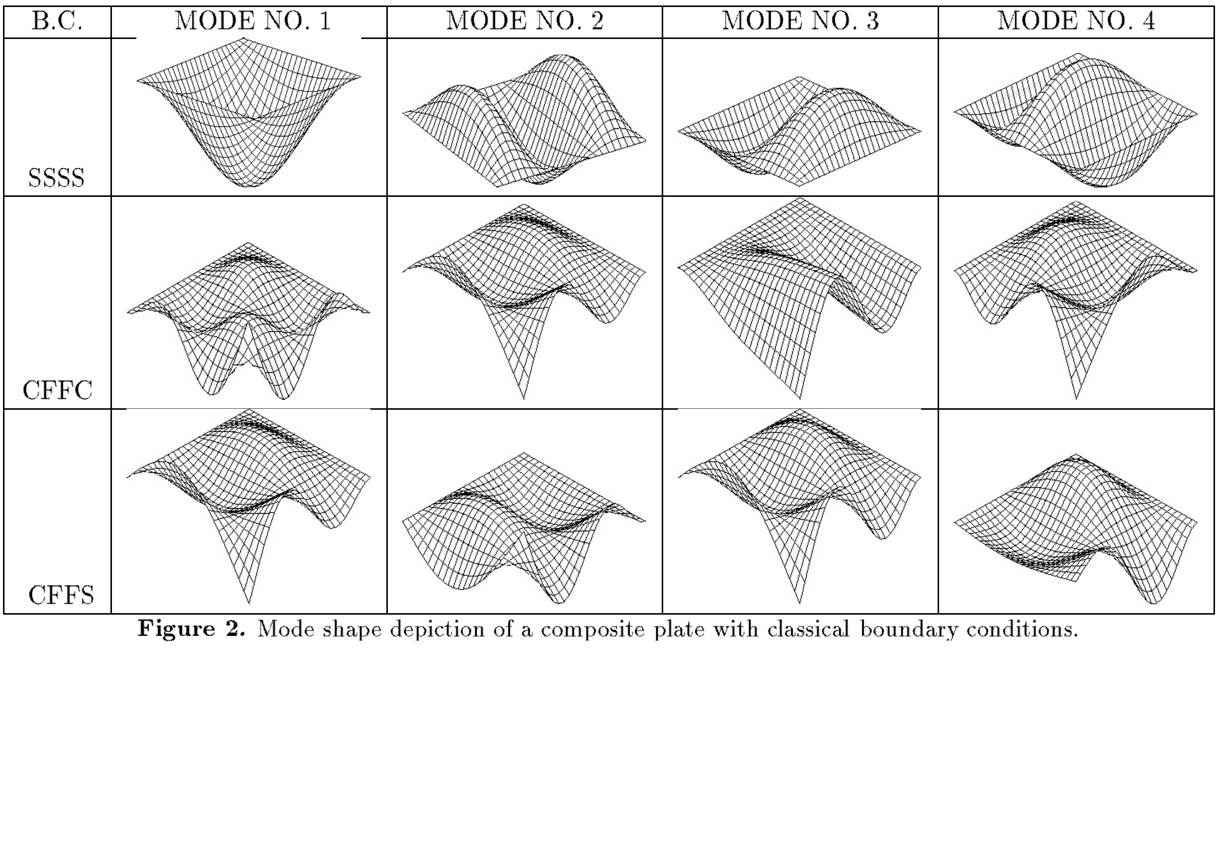

Figure 2. Mode shape depiction of a composite plate with classical boundary conditions.

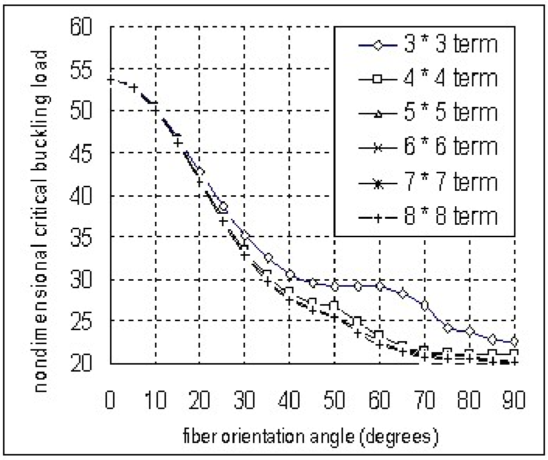

Figure 4. Convergence of non-dimensional critical Buckling load for a single layered composite plate with all edges clamped, (orthogonal polynomials).

devised in other works, but the whole application has remained obscure by evading a complete description of the procedure on how to construct the polynomials and by referring it to Bhat in the existing literature. Also, beam eigenfunctions have been applied using an interesting approach that is revealed to be superior when compared with the complex formulations devised in [5]. It has been verified that, as Bhat observed, the orthogonal polynomials yield superior results (lower quantities) for the lower modes, are simple to construct and possess the orthogonal property which simplifies the calculations. Using those try functions that have been provided by the Gram-Schmidt process defers the convergence of the R-R method to higher digits for M and N as supposed in the sigma formulae for deflection function (w) in the related equation in comparison with beam eigenfunctions. This is because of point that just the first member of the orthogonal polynomials set satisfies the natural boundary conditions where the other members satisfy only the geometrical boundary conditions (in classical case) or no boundary conditions (in elastic boundary conditions case) and this can be examined by a more careful look at the Gram-Schmidt formula. It has also been observed that the overall CPU time consumed to perform the calculations in the procedure to construct the orthogonal polynomials is about 30 times less than the corresponding time in the procedure to construct the approximating functions by using characteristic beam functions and it results from the simple nature of orthogonal polynomials. For validation of the results, the non-dimensional rigidities are set to large numbers as 10^8 or smallest quantity as zero that leads us to classical boundary conditions for a composite plate with more data available in the existing literature that helps us to rely on the computer code that was generated to perform the calculations. Also, the results have been compared with those gained by Rais –Rohani and Marcellier in [5].

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