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Stability Proof of Gain-Scheduling Controller for Skid-to-Turn Missile Using Kharitonov Theorem

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Gain scheduling is one of the most popular nonlinear control design approaches which has been widely and successfully applied in fields ranging from aerospace to process control. Despite the wide application of gain scheduling controllers, there is a notable lack of analysis on the stability of these controllers. The most common application of these kinds of controllers is in the field of flight control and autopilots. The main goal of this paper is to apply a methodology to prove stability of a gain scheduled controller used in directing Skid-to-Turn missiles.

One of the most widespread applications of gain scheduling controller is the main problem of this paper. To design the controller we use pole placement in state feedback controllers and a kind of innovative interpolation to reduce jumping in gains related to changing the flight conditions. Finally we utilize root locus and Kharitonov's Theorem to prove stability of the linearized plant. The presented approach for stability analysis is distinctive in the literature.

INTRODUCTION

Practical models of systems in engineering are often nonlinear and in many cases, plants have variable dynamics constrained in a specific operating region. In such cases, applying the concept of Gain Scheduling (GS) is a typical solution of the control problem [1, 2]. GS is one of the most popular approaches to non-linear, adaptive control systems design and has been the subject of extensive research over recent years, both from theoretical and practical viewpoints. This rehabilitated interest perhaps stems from the development of new techniques which allow a more systematic treatment of the GS problem.

Motion Control of flying objects may be the widest field of applying GS in practice. Design of longitudinal control system (autopilot) for a highly agile missile is a challenging problem, and has attracted the attention of many researchers [1, 3-7]. This is perhaps due to large variations in system parameters such as weight and moving method coupled with significant constraints on controller bandwidth. Although GS was introduced before two decades ago, the increasingly extension of its applications attracted researchers attention in 1990's. However, they did not pay enough attention to its theoretical analysis like stability proofs. Most of the papers were about presenting new interpolation methods or new applications of such a powerful method and at most analysis around these cases [3, 8-10].

The main contribution of this paper has two sections: The first one is introducing straight-forward design procedure for GS technique with an application in flight control of Skid-To-Turn (STT) Missiles. Secondly, a new approach is utilized to prove the stability of the closed loop controlled system. In stability analysis two different viewpoints are considered. Using poles' locations for linear system including interpolation led us to design better controllers but it can not be used for proof. Our approach to prove the stability was utilizing Kharitonov Theorem to cover the whole of operating space. This new point of view to a gain scheduling controller which was inspired from Robust Control was not existed in the literature of GS before.

The remaining of this paper is arranged as follows: In section II Gain Scheduling controller and its advantages and drawbacks are described. Stability and

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Kharitonov Theorem as our approach to this concept are explained in next section in detail. Dynamic of Skid-To-Turn Missile and designing controller description is the subject of Section IV. Section V describes the stability proof of the aforementioned controller. Finally section VI concludes this paper.

GAIN SCHEDULING

Gain Scheduling is one of the most applicable methods with a long usage period. Probably the most important motive in utilizing GS is existence of systems with variable dynamics. Absolutely it is one of the best methods of coping with variation of parameters in dynamic systems such as motion control of flying objects.

In this paper the focus is on gain scheduling in the sense of a controller with continuously varying coefficients according to the current value of scheduling signals. They are also called scheduling variables, that may be either exogenous signals or endogenous signals with respect to the plant. However, other interpretations are not completely ignored.

Designing Gain Scheduling controller can be explained in four steps. At first a linear parameter-varying model of the system should be determined. The most common approach is based on Jacobian linearization of the nonlinear plant about a family of equilibrium points (operating points). This yields a parameterized set of linearized plants and forms the basis for linearization scheduling. In fact these equations are functions of system variables and exogenous signals named “Scheduling Variables”. One of the most significant parts of designing GS controllers is selecting such scheduling variables that include the variations of the plant completely. Heuristically the variables with slower variations can be chosen as scheduling variables [1, 2].

The second step is to use linear design methods to design controllers for the linear parameter-varying plant model considering the control objects. This design process may result directly in a family of linear controllers corresponding to the linear parameter-dependent plant, or there may be an interpolation process to arrive at a family of linear controllers from a set of controller designs at the isolated values of the scheduling variables.

In the third step, which has the main role in the procedure, the set of designed controllers in operating points should be extended to controllers in whole of the state space. In other words this stage involves implementing the family of linear controllers such that the controller coefficients are scheduled according to the current value of the scheduling variables.

Finally, the fourth step is performance assessment. This includes the evaluation of implementing the final gain scheduled controller in nonlinear system. In the best case, where analytical performance guarantees are part of the design process, this may be relatively simple. More typically, the local stability and performance properties of the gain scheduled controller might be subject to analytical investigation, while the nonlocal performance evaluation is based on simulation studies [2].

Utilization of linear control designing methods in linearized models around scheduling points is the most important advantage of using GS to control nonlinear plants. This simplifies a complex nonlinear problem considerably. For example frequency domain methods and quadratic evaluation functions to investigate the performance and output/state feedback control are instances of common methods. Also applying robust control approaches especially when the model involves uncertainties is another aspect of GS. Such controllers are fast response to condition variation too [1].

To gain these advantages, the designer should cope with some difficulties too. One of these problems is determining the scheduling variables. As stated before selecting variables with slow variations is one way, but in practical applications, they are determined from the physical and implementing characteristics of the problem. It is not far from reality that is said “the function of scheduling variables that specifies the process of changing the gain is the most complexity of this method” [1, 9-11]. The common method is fitting a curve to existing data. Interpolation is another standard approach. In such conditions, it seems that using robust, multi variable or complex linear control methods converts GS to a more complicated technique. All in all, GS is not a general method and only includes a limited operating region [2, 9 and 10]. Off course it can be extended to consider whole of the required operating space with much effort. However, the vast usages of GS show that its advantages overcome the complexities and drawbacks.

STABILITY

Stability can be considered as a qualitative aspect of an engineering system. Obviously it is the most significant feature of a control system. All the practical and operating systems must have a stable design to work properly and usually an unstable system is useless and not applicable. The concept of stability for linear time invariant (LTI) systems is straightforward, but in dynamic nonlinear or time variant plants it will become more challenging [12].

Although there exist several techniques for stability proofs of linear systems, extension of stability analysis to nonlinear and time variant systems has a considerable importance in control engineering.

Kharitonov's theorem

In LTI systems the response to an input consists of a part similar to input and an exponential term. When the eigen-values of the system have negative real parts the exponential response tends to zero and the system is stable and vice versa (if the real parts were positive, the response will diverge unlimitedly and will become infinity; named unstable response) [12, 13]. LTI systems can be analyzed by calculating the characteristic equation and investigating its coefficients. But when the system changes related to time or parameters, the usual methods like Routh Hurwitz table are not applicable. In 1978 V. L. Kharitonov presented his theorem to solve this problem [14].

Kharitonov's Theorem: All polynomials of the form (1):

$$P(s,a) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + ... + a_n s^n \tag{1}$$

satisfying:

$$\underline{a_i} < a_i < \overline{a_i} \tag{2}$$

and

$$0 \notin [\underline{a_n}, \overline{a_n}] \tag{3}$$

are stable if and only if the following four polynomials are stable:

$$\begin{aligned} k_1(s,a) &= \underline{a_0} + \underline{a_1} s + \overline{a_2} s^2 + \overline{a_3} s^3 + \underline{a_4} s^4 + \underline{a_5} s^5 + ... \\ k_2(s,a) &= \overline{a_0} + \overline{a_1} s + \underline{a_2} s^2 + \underline{a_3} s^3 + \overline{a_4} s^4 + \overline{a_5} s^5 + ... \\ k_3(s,a) &= \overline{a_0} + \underline{a_1} s + \underline{a_2} s^2 + \overline{a_3} s^3 + \overline{a_4} s^4 + \underline{a_5} s^5 + ... \\ k_4(s,a) &= \underline{a_0} + \overline{a_1} s + \overline{a_2} s^2 + \underline{a_3} s^3 + \underline{a_4} s^4 + \overline{a_5} s^5 + ... \end{aligned} \tag{4}$$

In other words the coefficients of the polynomial can show a point in n dimensional space which are constrained in a n-cube determined by maximum and minimum values of parameters. Therefore the stability of Kharitonov's polynomials which are four of the end points in a n dimensional space guarantee the stability in whole of the constrained region inside the n-cube.

SKID-TO-TURN MISSILE AND DESIGNING THE CONTROLLER

As explained before, flying objects control is one of the applications that uses GS as a popular control method. In this section the formulation of STT missile is described. Also designed gain scheduled controller for its motion control is explained in the following.

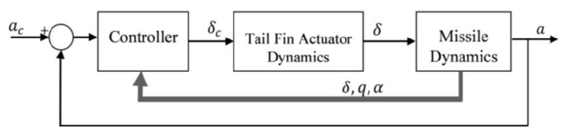

Figure 1. Block diagram of closed loop controlled system.

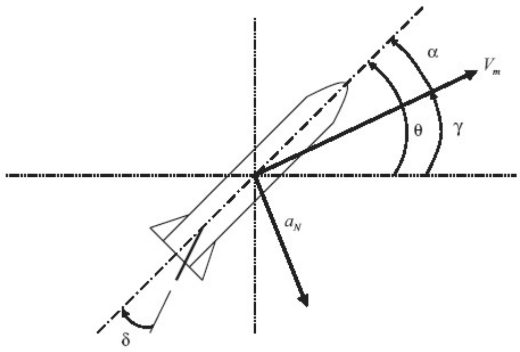

Figure 2. Dynamic model of the missile.

Skid-To-Turn missile dynamics

The missile dynamics considered here is taken from [15, 16]. The objective is to force a missile to track a desired given motion pattern. To this end, the guidance navigation system generates the reference acceleration commands for the center of the mass. This last one is compared to the actual normal acceleration measured in order to produce a tracking error. The control problem consists of generating a tail deflection (δ) that produces the angle of attack that corresponds to the required maneuver. The closed-loop block chart is shown in Figure 1.

To simplify the problem, the mass of the missile was considered invariant and it moved in one direction. With these simplifications, which are common in literature [4], the nonlinear motion equations summarized in two forces and one momentum. Through Figure 2 the formulation will be arranged as (5-7).

$$F_x = QSC_D \tag{5}$$

$$F_z = QSC_N \tag{6}$$

$$M_y = QSdC_M \tag{7}$$

Q is the dynamic pressure that depends on static pressure and relative velocity of the missile. Considering the angle of attack (α) and pitch rate (q) as the state variables, the differential equations and output equation of the system convert to (8)-(9) and (10) respectively;

$$\dot{\alpha} = \frac{\cos\alpha}{mV_m} F_z + q \tag{8}$$

$$\dot{q} = \frac{M_y}{I_y} \tag{9}$$

$$a_N = \frac{F_z}{m} \tag{10}$$

in which V_m denotes the missile speed and C_N and C_M are the aerodynamic coefficients which are described as below:

$$C_D(\alpha, \delta, M) = -0.3 \tag{11}$$

$$C_N(\alpha, \delta, M) = \beta_{1N}\alpha^3 + \beta_{2N}\alpha\,|\alpha| + \beta_{3N}\left(2 - \frac{M}{3}\right)\alpha + d_n\delta$$
$$= c_n(\alpha, M) + d_n\delta \tag{12}$$

$$C_M(\alpha, \delta, M) = \beta_{1M}\alpha^3 + \beta_{2M}\alpha\,|\alpha| + \beta_{3M}\left(-7 + \frac{8M}{3}\right)\alpha$$
$$+ d_m\delta = c_m(\alpha, M) + d_m\delta \tag{13}$$

where β_{iN}, β_{iM}, d_n and d_m are the constant aerodynamic polynomial coefficients.

Tail deflection actuator has the following dynamics:

$$\dot{\delta} = \frac{1}{\tau_a}(\delta_c - \delta) \tag{14}$$

In this formula δ_c is the commanded tail deflection which is produced by controller, and tail deflection is limited to $|\delta| \leq 40°$.

Designing gain scheduling feedback controllers

To design a desired Gain-Scheduling controller, at first a suitable system model which is the function of system parameters should be prepared. The next step in designing GS controllers is defining appropriate scheduling variables. Angle of attack and Mach number are to be used for scheduling purpose. This is mainly due to the fact that these two variables are best in revealing dynamical changes in the system and can be estimated with no difficulty. Therefore, finding a linear parameter variable (LPV) model based on these variables, is the aim of the first stage. This attitude resulted in nonlinear state and output equations as follows.

$$\dot{\alpha} = k_\alpha M C_N \cos\alpha + q, \qquad k_\alpha = \frac{0.7P_0S}{mV_s} \tag{15}$$

$$\dot{q} = k_q M^2 C_M, \qquad k_q = \frac{0.7P_0Sd}{I_y} \tag{16}$$

$$a_N = k_z C_N M^2, \qquad k_z = \frac{0.7P_0S}{m} \tag{17}$$

As mentioned before one of the advantages of GS controllers is the ability of utilizing linear methods for designing controllers. Thus, linearization of the equations around equilibrium points and using linear pole placement result in the desired performance. With this approach the linear system equations can be calculated according to (18).

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Aa & 1 \\ Ma & 1 \end{bmatrix} \begin{bmatrix} \alpha_\delta \\ q_\delta \end{bmatrix} + \begin{bmatrix} B_d \\ M_d \end{bmatrix} \delta \tag{18}$$

The output equation will be as (19):

$$a_N = \begin{bmatrix} C_{a_N} & 0 \end{bmatrix} \begin{bmatrix} \alpha_\delta \end{bmatrix} + d_d\delta \tag{19}$$

and the coefficients formulations are described by (20)-(25).

$$A_a = K_\alpha M \left[\frac{\partial c_n}{\partial \alpha}\cos\alpha - \left(c_n - \frac{d_n c_m}{d_m}\right)\sin\alpha\right] \tag{20}$$

$$M_a = K_q M^2 \frac{\partial c_m}{\partial \alpha} \tag{21}$$

$$B_d = K_\alpha M d_n \cos\alpha \tag{22}$$

$$M_d = K_q M^2 d_m \tag{23}$$

$$d_d = K_z M^2 d_n \tag{24}$$

$$C_{a_N} = K_z M^2 \frac{\partial c_n}{\partial \alpha} \tag{25}$$

The variables of the linear model are α and M which are the scheduling variables. On the other hand, α is a system state variable too. Such systems are named Quassi LPV (QLPV). But the significant problem is that this parameter is not measurable and the only measurable variables are a_N and q. This problem can be solved via a simple parameter changing that introduces a new state variable:

$$Z(t) = a_N^{com} - a_N \tag{26}$$

With these modifications the formulations are converted to a different new appearance. Assume that the new state variables vector is $x = \begin{bmatrix} z & q & \delta & \int z \end{bmatrix}^T$. New equations will be computed as (27) -(29):

$$\dot{x} = Ax + B\delta_c + Ea_N^{com} \tag{27}$$

$$A = \begin{bmatrix} A_a & -c_{a_N} & \frac{d_d}{\tau_a} - B_d C_{a_N} + A_a d_d & 0 \\ \frac{-M_a}{C_{a_N}} & 0 & M_d - \frac{M_a d_d}{C_{a_N}} & 0 \\ 0 & 0 & \frac{-1}{\tau_a} & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

$$B = \begin{bmatrix} \frac{-d_d}{\tau_a} \\ 0 \\ \frac{1}{\tau_a} \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} -A_a \\ \frac{M_a}{C_{a_n}} \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

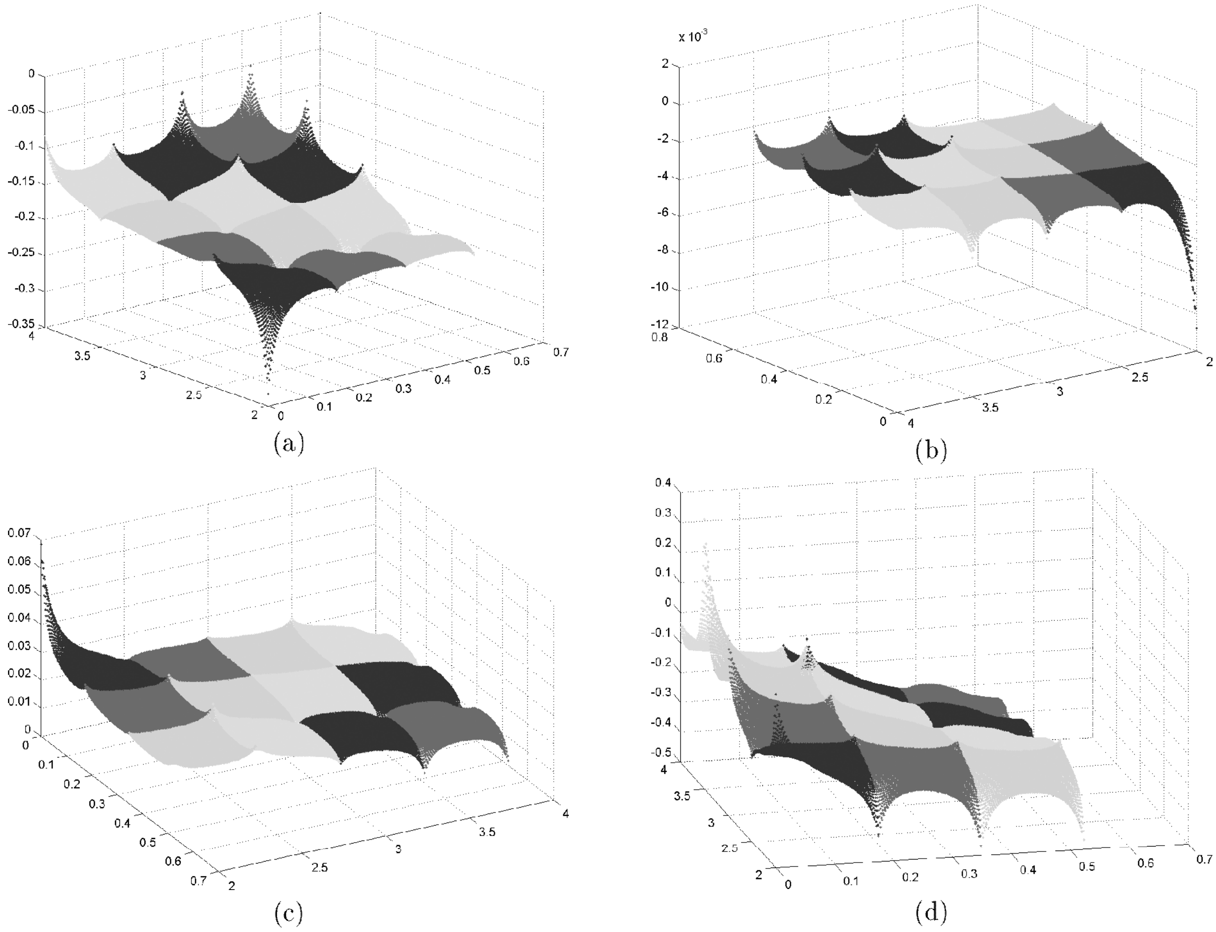

Figure 3. The controllers gains changes related to Mach number (X-Axis) and angle of attack (Y-Axis).

Now, a proper model of system is ready to use for designing the controller. In this step, using the pole placement and state feedback, desired controllers were designed in twenty operating points as scheduling points. The number of points is chosen experimentally and like that Mehrabian et al selected [4]. In the next step, to compute controller coefficients, linear interpolation between neighbor points gives the coefficients of controller in plates between them. Considering whole the points' effect proportional to distances between this point and the other ones determines the required controller. Therefore; the changes in each point affect on the coefficients of the other points and alter the plates between the points. Obviously, the points near each operating point are influenced mostly by that point. Figure 3 shows the plates that are constructed between operating points for four coefficients of the states.

The performance of resulted GS controller is depicted in Figure 4. The input signal tracking is desirable and step-like variations are followed by minimum perturbation. The abrupt variations in input signal do not resulted in sudden variations and large values of control effort derivative. Also desired time constant and overshoot were considered in designing the linear controllers in each point.

STABILITY ANALYSIS OF THE CONTROLLED CLOSED LOOP SYSTEM

Root Locus Approach

The GS controller which was described in the previous section has a satisfying performance in simulations and keeps the stability of the system. In this section the stability is investigated via a basic concept in linear systems stability analysis, root locus. In linear systems, everything about stability can be realized from locations of the systems' poles; systems that have poles with negative real parts (poles place in the side of imaginary axis) are stable and even only one pole with positive real part makes it unstable. Using this simple concept, the stability of the closed loop including GS controller and nonlinear model of missile can be analyzed.

Figure 5 shows the poles of the closed loop system that contains the linearized system and controller in

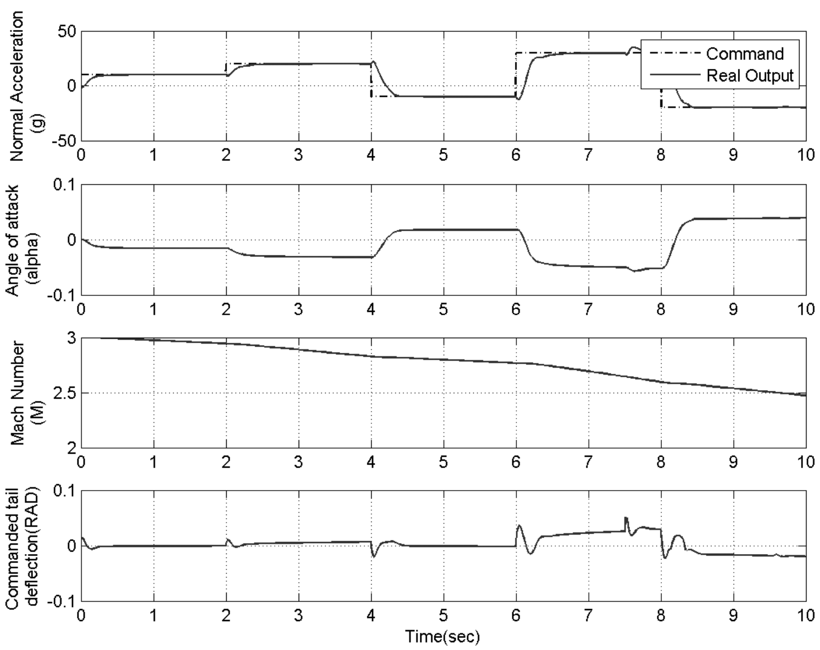

Figure 4. Performance of the system responding to changing step-like input.

numerous points covering the space. Obviously the number of points is limited and between each two pints there exist infinite points which are not evaluated. Therefore; although this cannot prove the stability, at least being all the points in right half plane, as illustrated in Figure 5, can increase the confidence of being stable in the whole of the space. For this reason another approach was required to prove the stability which persuaded us to use Kharitonov Theorem to cover the whole space.

The New Approach; Using Kharitonov Theorem

As described before, root locus drawn in Figure 5 shows the behavior of system poles through possible variations and can be used to ensure us about the stability. On the other hand, if this figure contains only one point in the right half plane, then it concludes that the system is unstable. However; this approach is not an acceptable stability proof. Thus Kharitonov theorem is utilized in this section to present a complete proof as a new point of view in such problems.

The aforementioned GS controller can be assumed as a state feedback controller which its gains are tuned via pole placement. The system characteristic equation will be easily computed using gain scheduled state feedback controller and system model. Variations of the system parameters besides the changing controller result in characteristic equations with variable coefficients. Also because the limits of the variations of

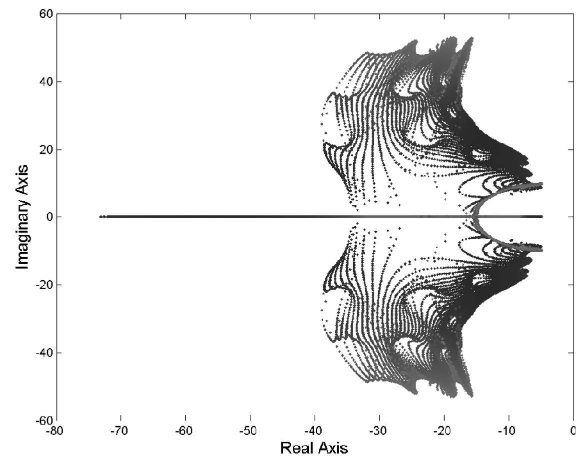

Figure 5. The closed loop root locus. The scheduling parameters are changing with tiny variations to stabilize the whole of the possible conditions reasonably.

the system parameters in aforementioned plates are determined, the controller gains and model variation limits can be calculated easily. Then using Kharitonov theorem, computing related four polynomials in each plate and investigating their stability the proof will be completed. Table 1 shows these limits and the polynomials are calculated via (4). From this point of view stability will be guaranteed in each plate and because of the continuity between the plates the proof will be generalized to the whole of the space.

CONCLUSION

A new approach about stability analysis of gain scheduling controllers for a specific kind of missiles named Skid-to-Turn missile was presented in this paper. At first, to apply the gain scheduling as a controller designing method, a new interpolation was presented which was used in the motion control of the missile. The stability of the resulted controller was illustrated with root locus of the closed loop linearized system.

Finally by checking Kharitonov polynomials in operating points, where the controllers were designed, the stability was proven in the whole of the operating spaces. Using such approach for proving the stability of gain scheduling feedback controller is an original point of view to this problem presented in this paper uniquely. These can be utilized as a useful tool to prove similar scenarios.

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Table 1. Range of variations of the coefficients related to state changes for controlled system.

α(rad) / M	$0 \leq \alpha \leq 10$	$10 \leq \alpha \leq 20$	$20 \leq \alpha \leq 30$
$2 \leq M \leq 2.5$	$10436 < a_0 < 248780$ $19790 < a_1 < 46360$ $1750 < a_2 < 3050$ $70 < a_3 < 90$	$114140 < a_0 < 315350$ $23530 < a_1 < 55400$ $1770 < a_2 < 3280$ $70 < a_3 < 90$	$121530 < a_0 < 337210$ $23360 < a_1 < 50100$ $1790 < a_2 < 3480$ $70 < a_3 < 100$
$2.5 \leq M \leq 3$	$113930 < a_0 < 383220$ $24240 < a_1 < 62990$ $2040 < a_2 < 371$ $70 < a_3 < 100$	$405830 < a_0 < 240430$ $74880 < a_1 < 47480$ $2710 < a_2 < 3820$ $90 < a_3 < 100$	$269700 < a_0 < 425120$ $54360 < a_1 < 78420$ $2700 < a_2 < 3480$ $80 < a_3 < 100$
$3 \leq M \leq 3.5$	$265840 < a_0 < 565160$ $35800 < a_1 < 84990$ $3140 < a_2 < 4460$ $80 < a_3 < 110$	$346280 < a_0 < 595390$ $60830 < a_1 < 100420$ $3460 < a_2 < 4570$ $80 < a_3 < 110$	$595250 < a_0 < 263000$ $46780 < a_1 < 102010$ $2690 < a_2 < 4020$ $80 < a_3 < 100$
$3.5 \leq M \leq 4$	$165830 < a_0 < 661760$ $34460 < a_1 < 101930$ $2520 < a_2 < 4670$ $70 < a_3 < 110$	$333820 < a_0 < 804720$ $59540 < a_1 < 126540$ $3210 < a_2 < 4810$ $70 < a_3 < 110$	$170500 < a_0 < 525970$ $35910 < a_1 < 131270$ $2690 < a_2 < 4590$ $70 < a_3 < 100$

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