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Differential Equation of Sensitivity Matrix for Final Velocity Constraint

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In this paper, differential equation of sensitivity matrix of required velocity with respect to position vector is derived for a linearized problem of a spacecraft motion with final velocity constraint. This matrix can be utilized for an implicit guidance of a spacecraft with final position and velocity constraint. The solution is obtained in the presence of aerodynamical force. Therefore, the resulting equation can be used for trajectory optimization. Moreover, a change of variable is presented to reduce onboard computational burden. In addition, the method is more appropriate for analytical approaches than conventional implicit equations.

INTRODUCTION

Guidance laws based on velocity-to-be-gained may be classified into two categories, namely explicit and implicit schemes [1,2]. In explicit guidance, the required velocity is computed onboard explicitly whereas the velocity-to-be-gained is computed via a first-order differential equation in the well-known Q guidance as an implicit one. In Q guidance variants, a required velocity is defined as the instantaneous velocity required to satisfy a set of final objectives [3-6]. Extensive research has been done for implementation of Q guidance by many investigators, from 1960's up to now, some of which can be found in Refs. [5-10].

In 1987, Bhat and Shrivastava developed a modified Q-guidance scheme for placing a payload into a specified circular orbit [11]. An implicit guidance equation has been added to the well-known Q-guidance scheme in order to modify implicit guidance equations for orbit injection, *i.e.*, it can be utilized for final position and velocity constraints [12]. This added equation is based on the sensitivity matrix of required velocity with respect to position vector when final velocity is constrained, denoted by Q_v.

It is worth noting that the velocity-to-be-gained guidance technique presented in Ref. [4] is workable if it is possible to define, at each instant of thrusting, a required velocity to meet mission objectives which is a function only of current position, as stated by Battin. This requirement cannot be met for the problem having both final position and velocity constraints.

There is another type of guidance law that determines a near optimal or an effective direction of the thrust vector for orbit injection [13-17]. In this class of guidance scheme, an explicit or iterative [17] algorithm is utilized for calculation of the thrust direction. Implicit and explicit guidance algorithms have their advantages and disadvantages which are beyond the scope of this paper.

In the present work, the differential equation of Q_v is derived. This sensitivity matrix is utilized for a modified implicit guidance with final position and velocity constraints. For the purpose of trajectory optimization, the equations are obtained in the presence of atmosphere. Finally, a change of variable is proposed for conventional and modified implicit guidance schemes.

REQUIRED VELOCITY

The governing equation of motion of a vehicle as a particle is, here, modeled by a linear time-varying differential equation as follows:

$$\ddot{\mathbf{r}} = \mathbf{f}(t) + F_r(t)\mathbf{r} + F_v(t)\mathbf{v} + \mathbf{a}_c \tag{1}$$

where $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}_c$ are the vehicle position, velocity, and achieved commanded acceleration vectors with respect to an inertial reference, respectively. Also, $\mathbf{f}(t)$ is a 3×1 vector; $F_r(t)$ and $F_v(t)$ are 3×3 matrices, that may be obtained by linearization about a reference trajectory. The achieved commanded acceleration depends on

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control system dynamics. The system dynamics is modeled by the following state space representation:

$$\dot{\mathbf{r}} = \mathbf{v} \tag{2a}$$

$$\dot{\mathbf{v}} = \mathbf{f}(t) + F_r(t)\mathbf{r} + F_v(t)\mathbf{v} + \mathbf{a}_c \tag{2b}$$

$$\dot{\mathbf{a}}_c = A^a(t)\mathbf{x} + B_a(\mathbf{x},\mathbf{u},t)\mathbf{u} \tag{2c}$$

$$\dot{\mathbf{p}} = A^p(t)\mathbf{x} + B_p(\mathbf{x},\mathbf{u},t)\mathbf{u} \tag{2d}$$

where $\mathbf{x} = [\mathbf{r}^T\ \mathbf{v}^T\ \mathbf{a}_c^T\ \mathbf{p}^T]^T$ is the state vector; $\mathbf{p}$ is an arbitrary n-component vector including remaining state variables of a control system; $\mathbf{u}$ is the commanded acceleration (control input); $A^a(t)$ and $A^p(t)$ are submatrices of the system matrix; and $B_a(\mathbf{x},\mathbf{u},t)$ and $B_p(\mathbf{x},\mathbf{u},t)$ are analytical functions in our domain of interest. The superscript "T" represents the transpose of a vector or matrix. All matrices are of appropriate dimensions.

The system matrix and its fundamental matrix, if obtainable, are written as

$$A(t) = \begin{bmatrix} 0 & I & 0 & 0 \\ F_r(t) & F_v(t) & I & 0 \\ A_r(t) & A_v(t) & A_a(t) & A_p(t) \\ P_r(t) & P_v(t) & P_a(t) & P_p(t) \end{bmatrix} \tag{3a}$$

$$A^a = [A_r\ A_v\ A_a\ A_p] \tag{3b}$$

$$A^p = [P_r\ P_v\ P_a\ P_p] \tag{3c}$$

$$\Phi(t,t_0) = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} \end{bmatrix} \tag{4}$$

where 0 and I are zero and identity matrices of appropriate dimensions, respectively. The argument of Φ_{ij} in the preceding relation is (t, t_0), not shown for compactness.

Equation (1) can be rewritten in the following matrix form:

$$\frac{d}{dt}\begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ F_r(t) & F_v(t) \end{bmatrix}\begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{a}_c + \mathbf{f}(t) \end{bmatrix} \tag{5}$$

Consider $\Phi^R(t,t_0)$ as the fundamental matrix for the following system matrix:

$$A^R(t) = \begin{bmatrix} 0 & I \\ F_r(t) & F_v(t) \end{bmatrix} \tag{6}$$

The fundamental matrix $\Phi^R(t,t_0)$ is partitioned into four 3×3 submatrices, that is,

$$\Phi^R(t,t_0) = \begin{bmatrix} \Phi_{11}^R(t,t_0) & \Phi_{12}^R(t,t_0) \\ \Phi_{21}^R(t,t_0) & \Phi_{22}^R(t,t_0) \end{bmatrix} \tag{7}$$

Consider a linear differential equation of:

$$\dot{\mathbf{z}} = A^R(t)\mathbf{z} + \mathbf{G}(\mathbf{x},\mathbf{u},t) \tag{8}$$

which integrates into:

$$\mathbf{z}(t) = \Phi^R(t,t_0)\mathbf{z}(t_0) + \int_{t_0}^{t} \Phi^R(t,\lambda)\mathbf{G}(\mathbf{x}(\lambda),\mathbf{u}(\lambda),\lambda)d\lambda \tag{9}$$

where $\mathbf{z} = [\mathbf{r}^T\ \mathbf{v}^T]^T$ and:

$$\mathbf{G}(\mathbf{x},\mathbf{u},t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{a}_c + \mathbf{f}(t) \end{bmatrix} \tag{10}$$

Substitution yields:

$$\mathbf{r}(t) = \Phi_{11}^R(t,t_0)\mathbf{r}(t_0) + \Phi_{12}^R(t,t_0)\mathbf{v}(t_0) + \int_{t_0}^{t} \Phi_{12}^R(t,\lambda)[\mathbf{f}(\lambda) + \mathbf{a}_c(\lambda)]d\lambda \tag{11}$$

$$\mathbf{v}(t) = \Phi_{21}^R(t,t_0)\mathbf{r}(t_0) + \Phi_{22}^R(t,t_0)\mathbf{v}(t_0) + \int_{t_0}^{t} \Phi_{22}^R(t,\lambda)[\mathbf{f}(\lambda) + \mathbf{a}_c(\lambda)]d\lambda \tag{12}$$

The preceding relation is rewritten between a specified final time t_f and the current time t as follows:

$$\mathbf{v}(t_f) = \Phi_{21}^R(t_f,t)\mathbf{r}(t) + \Phi_{22}^R(t_f,t)\mathbf{v}(t) + \int_{t}^{t_f} \Phi_{22}^R(t_f,\lambda)[\mathbf{f}(\lambda) + \mathbf{a}_c(\lambda)]d\lambda \tag{13}$$

On the other hand, the achieved commanded acceleration without input control $\mathbf{u}$ is found to be:

$$\mathbf{a}_c(t) = \Phi_{31}(t,t_0)\mathbf{r}(t_0) + \Phi_{32}(t,t_0)\mathbf{v}(t_0) + \Phi_{33}(t,t_0)\mathbf{a}_c(t_0) + \Phi_{34}(t,t_0)\mathbf{p}(t_0) + \int_{t_0}^{t} \Phi_{32}(t,\xi)\mathbf{f}(\xi)d\xi \tag{14}$$

Therefore, for $t \le \lambda \le t_f$ we arrive at:

$$\mathbf{a}_c(\lambda) = \Phi_{31}(\lambda,t)\mathbf{r}(t) + \Phi_{32}(\lambda,t)\mathbf{v}(t) + \Phi_{33}(\lambda,t)\mathbf{a}_c(t) + \Phi_{34}(\lambda,t)\mathbf{p}(t) + \int_{t}^{\lambda} \Phi_{32}(\lambda,\xi)\mathbf{f}(\xi)d\xi \tag{15}$$

The final velocity without control effort is then given by:

$$\mathbf{v}(t_f) = \Phi_{21}^R(t_f,t)\mathbf{r}(t) + \Phi_{22}^R(t_f,t)\mathbf{v}(t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\mathbf{f}(\lambda)d\lambda + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)[\Phi_{31}(\lambda,t)\mathbf{r}(t) + \Phi_{32}(\lambda,t)\mathbf{v}(t) + \Phi_{33}(\lambda,t)\mathbf{a}_c(t) + \Phi_{34}(\lambda,t)\mathbf{p}(t)]d\lambda + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\left[\int_t^{\lambda} \Phi_{32}(\lambda,\xi)\mathbf{f}(\xi)d\xi\right]d\lambda \quad (16)$$

Rearrangement gives:

$$\mathbf{v}(t_f) = \left[\Phi_{21}^R(t_f,t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{31}(\lambda,t)d\lambda\right]\mathbf{r}(t) + \left[\Phi_{22}^R(t_f,t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{32}(\lambda,t)d\lambda\right]\mathbf{v}(t) + \left[\int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{33}(\lambda,t)d\lambda\right]\mathbf{a}_c(t) + \left[\int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{34}(\lambda,t)d\lambda\right]\mathbf{p}(t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\mathbf{f}(\lambda)d\lambda + \int_t^{t_f}\int_t^{\lambda} \Phi_{22}^R(t_f,\lambda)\Phi_{32}(\lambda,\xi)\mathbf{f}(\xi)d\xi d\lambda \quad (17)$$

The desired velocity $\mathbf{v}_v^*(t)$ is defined as an instantaneous velocity, required to satisfy the final velocity constraint $\mathbf{v}^*(t_f)$ without any control effort. Hence:

$$\mathbf{v}^*(t_f) = \left[\Phi_{21}^R(t_f,t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{31}(\lambda,t)d\lambda\right]\mathbf{r}(t) + \left[\Phi_{22}^R(t_f,t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{32}(\lambda,t)d\lambda\right]\mathbf{v}_v^*(t) + \left[\int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{33}(\lambda,t)d\lambda\right]\mathbf{a}_c(t) + \left[\int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{34}(\lambda,t)d\lambda\right]\mathbf{p}(t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\mathbf{f}(\lambda)d\lambda + \int_t^{t_f}\int_t^{\lambda} \Phi_{22}^R(t_f,\lambda)\Phi_{32}(\lambda,\xi)\mathbf{f}(\xi)d\xi d\lambda \quad (18)$$

Solving for the desired velocity $\mathbf{v}_v^*(t)$, the following expression will be obtained if the second item bracketed in Eq. (18) is invertible:

$$\mathbf{v}_v^* = \left[\Phi_{22}^R(t_f,t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{32}(\lambda,t)d\lambda\right]^{-1} \left\{\mathbf{v}^*(t_f) - \left[\Phi_{21}^R(t_f,t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{31}(\lambda,t)d\lambda\right]\mathbf{r} - \left[\int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{33}(\lambda,t)d\lambda\right]\mathbf{a}_c - \left[\int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{34}(\lambda,t)d\lambda\right]\mathbf{p} - \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\mathbf{f}(\lambda)d\lambda - \int_t^{t_f}\int_t^{\lambda} \Phi_{22}^R(t_f,\lambda)\Phi_{32}(\lambda,\xi)\mathbf{f}(\xi)d\xi d\lambda\right\} \quad (19)$$

Taking the partial derivative with respect to $\mathbf{r}$ results in the sensitivity matrix for the linearized problem, that is:

$$\frac{\partial \mathbf{v}_v^*}{\partial \mathbf{r}} = -\left[\Phi_{22}^R(t_f,t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{32}(\lambda,t)d\lambda\right]^{-1} \times \left[\Phi_{21}^R(t_f,t) + \int_t^{t_f} \Phi_{22}^R(t_f,\lambda)\Phi_{31}(\lambda,t)d\lambda\right] \quad (20)$$

The preceding relation could have also been obtained in the form of $\Phi_{22}^{-1}(t_f,t)\Phi_{21}(t_f,t)$. However, one advantage of the presented method is that Eq. (20) is derived in terms of Φ_{31} and Φ_{32}, indicating the dependence of $\mathbf{a}_c$ on position and velocity.

Now consider the case in which the state variables of control system, *i.e.*, $\mathbf{a}_c$ and $\mathbf{p}$ are not dependent on the vehicle position and velocity. In this case, the system matrix (3a) simplifies to:

$$A(t) = \begin{bmatrix} 0 & I & 0 & 0 \\ F_r(t) & F_v(t) & I & 0 \\ 0 & 0 & A_a(t) & A_p(t) \\ 0 & 0 & P_a(t) & P_p(t) \end{bmatrix} \quad (21)$$

The control system can also be expressed as:

$$\begin{bmatrix} \dot{\mathbf{a}}_c \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} A_a(t) & A_p(t) \\ P_a(t) & P_p(t) \end{bmatrix} \begin{bmatrix} \mathbf{a}_c \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} B_1(\mathbf{a}_c,\mathbf{p},\mathbf{u},t) \\ B_2(\mathbf{a}_c,\mathbf{p},\mathbf{u},t) \end{bmatrix}\mathbf{u} \quad (22)$$

For this case, we have $\Phi_{31}(t,t_0) = 0$, $\Phi_{32}(t,t_0) = 0$, $\Phi_{41}(t,t_0) = 0$, and $\Phi_{42}(t,t_0) = 0$. Hence, Eq. (20) simplifies to:

$$\frac{\partial \mathbf{v}_v^*}{\partial \mathbf{r}} = -\Phi_{22}^{R^{-1}}(t_f,t)\Phi_{21}^R(t_f,t) \quad (23)$$

DIFFERENTIAL EQUATION OF SENSITIVITY MATRIX

Consider the system matrix of a system to be given by Eq. (21), that is:

$$A(t)=\begin{bmatrix}0 & I & 0 & 0\\ F_r(t) & F_v(t) & I & 0\\ 0 & 0 & A_a(t) & A_p(t)\\ 0 & 0 & P_a(t) & P_p(t)\end{bmatrix} \tag{24}$$

Now define the matrix $C(t_f,t)$ as follows:

$$C(t_f,t)=-\Phi_{22}^{R^{-1}}(t_f,t)\Phi_{21}^{R}(t_f,t) \tag{25}$$

Differentiation yields:

$$\dot{C}(t_f,t)=-\dot{\Phi}_{22}^{R^{-1}}(t_f,t)\Phi_{21}^{R}(t_f,t) -\Phi_{22}^{R^{-1}}(t_f,t)\dot{\Phi}_{21}^{R}(t_f,t) \tag{26}$$

Using the relation $d(M^{-1})/dt=-M^{-1}\dot{M}M^{-1}$ for the differentiation of an invertible square matrix M, we have:

$$\frac{d}{dt}\Phi_{22}^{R^{-1}}(t_f,t)=-\Phi_{22}^{R^{-1}}(t_f,t)\dot{\Phi}_{22}^{R}(t_f,t)\Phi_{22}^{R^{-1}}(t_f,t) \tag{27}$$

Using the property of $\frac{d}{dt}\Phi^R(t_1,t)=-\Phi^R(t_1,t)A^R(t)$ we obtain:

$$\dot{\Phi}_{21}^{R}(t_f,t)=-\Phi_{22}^{R}(t_f,t)F_r(t) \tag{28}$$

Substitution of Eqs. (27) and (28) into Eq. (26) results in:

$$\dot{C}(t_f,t)-F_v(t)C(t_f,t)+C^2(t_f,t)=F_r(t) \tag{29}$$

The preceding differential equation for: $C(t_f,t)=-\Phi_{22}^{R^{-1}}(t_f,t)\Phi_{21}^{R}(t_f,t)$ will be obtained if the system matrix is in the form of Eq. (24).

From Eq. (23) it turns out for system (2) with the system matrix (24), that the partial derivative of the desired velocity $\mathbf{v}_v^*$ with respect to $\mathbf{r}$, *i.e.*, $\partial\mathbf{v}_v^*/\partial\mathbf{r}=Q_v$ is equal to $C(t_f,t)$. Hence:

$$\dot{Q}_v-F_v(t)Q_v+Q_v^2=F_r(t) \tag{30}$$

To calculate Q_v from the preceding differential equation, the initial or final value of Q_v is needed. From definition (25) we have:

$$C(t_f,t_1)=-\Phi_{22}^{R^{-1}}(t_f,t_1)\Phi_{21}^{R}(t_f,t_1) \tag{31}$$

From the property of $\Phi^R(t_1,t_1)=I$ we have:

$$\Phi_{11}^{R}(t_1,t_1)=I_{3\times3} \tag{32a}$$

$$\Phi_{22}^{R}(t_1,t_1)=I_{3\times3} \tag{32b}$$

$$\Phi_{12}^{R}(t_1,t_1)=0_{3\times3} \tag{32c}$$

$$\Phi_{21}^{R}(t_1,t_1)=0_{3\times3} \tag{32d}$$

It follows that the final value of $C(t_f,t)$ can be calculated. Thus:

$$C(t_f,t_f)=-\Phi_{22}^{R^{-1}}(t_f,t_f)\Phi_{21}^{R}(t_f,t_f)=0_{3\times3} \tag{33}$$

Hence, the final value of Q_v becomes $0_{3\times3}$. Note that the differential equation of Q_v has been obtained here assuming the state variables of control system, *i.e.* $\mathbf{a}_c$ and $\mathbf{p}$, are not dependent on the vehicle position and velocity.

The differential equation of Q_v can be reduced to:

$$\dot{Q}_v+Q_v^2=F_r(t) \tag{34}$$

provided that the cut off (or burnout) occurs in the exoatmosphere. In this case, the desired velocity reduces to a hypothetical velocity referred to as "required velocity" defined in the vacuum. As a simple approximation, neglecting Q_v^2 in comparison to $F_r(t)$, and integrating results in:

$$Q_v(t_f,t_f)-Q_v(t_f,t)=\int_t^{t_f}F_r(\xi)d\xi \tag{35}$$

Applying $Q_v(t_f,t_f)=0_{3\times3}$ yields:

$$Q_v(t_f,t)\approx-\int_t^{t_f}F_r(\xi)d\xi \tag{36}$$

Now, a vertical planar motion in the spherical Earth model is considered. The vehicle nominal trajectory is given by:

$$x_I=R_e\cos(\pi/3)+4.73t^2 \tag{37a}$$

$$z_I=R_e\sin(\pi/3)+8.87t^2 \tag{37b}$$

where R_e is the Earth radius and (x_I,z_I) are the Earth-centered inertial coordinates.

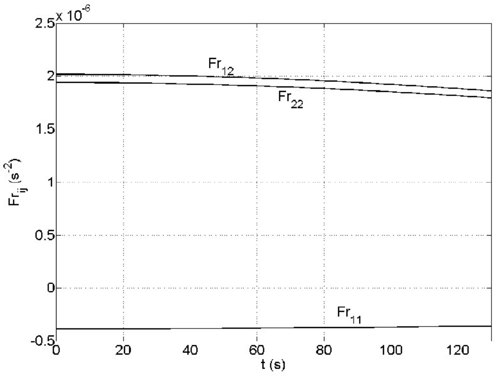

Figure 1. Elements of F_r matrix versus time.

Figure 1 shows the behavior of $F_r = \partial\mathbf{g}/\partial\mathbf{r}$ elements versus time for the nominal trajectory. Approximate solution (36) may further be simplified to:

$$Q_v(t_f, t) \approx -t_{go}F_r(t) = \frac{\mu t_{go}}{r_I^5}(r_I^2 I_{3\times3} - \mathbf{r}_I\mathbf{r}_I^T) \qquad (38)$$

where $t_{go} = t_f - t$ is the time-to-go until the final time, $\mathbf{r}_I$ is the vehicle position vector ($r_I = |\mathbf{r_I}|$), and μ is the Earth gravitational parameter. In a planar motion, it is easy to show that Q_v matrix obtained from Eq. (34) will be symmetrical one when $\partial\mathbf{g}/\partial\mathbf{r}$ is symmetrical.

Figure 2 depicts the three different solution methods for Q_v elements. In this figure, the numerical solution of differential equation of Q_v for the linearized problem is shown by solid lines, approximate solution (38) by dash-lines, and the approximate solution of $\Delta\mathbf{v}_v^*/\Delta\mathbf{r}$ using a nonlinear flight simulation by circles. The results are case dependent, and approximate solution (38) needs to be modified.

As a future study, the exact analytical solution for Q_v such as the one developed by Martin [5] for Q_p, is suggested to be done. Also, perturbation techniques may give accurate approximation for onboard calculations. As a comparative work, the methods of calculations of Q_v and their accuracies can be compared.

REDUCED FORM

The differential equation of the velocity-to-be-gained, the required velocity minus the current velocity of a spacecraft, is given by [3]:

$$\frac{d\mathbf{v}_g}{dt} = -Q\mathbf{v}_g - \mathbf{a}_T \qquad (39)$$

where $\mathbf{a}_T$ is the nongravitational acceleration. Since an accelerometer can only sense the resultant nongravitational acceleration, the required velocity is defined for the vacuum. Therefore, Q matrix is calculated in the vacuum. To calculate the two velocity errors for final position and velocity constraints, we have [3,12]:

$$\frac{d\mathbf{v}_g^i}{dt} = -Q_i\mathbf{v}_g^i - \mathbf{a}_T\ , \quad i = p \text{ or } v \qquad (40)$$

where the subscript or superscript "p" deduces conventional Q guidance, and the subscript or superscript "v" deduces the differential equation of the velocity-to-be-gained for final velocity constraint ($\mathbf{v}_g^v = \mathbf{v}_v^* - \mathbf{v}$).

Using the following change of variable:

$$\mathbf{z}_i = F_i\,\mathbf{v}_g^i \qquad (41)$$

we have $\dot{\mathbf{z}}_i = \dot{F}_i\mathbf{v}_g^i + F_i\dot{\mathbf{v}}_g^i$. The present change of variable was introduced in Ref. [18], and here it is utilized for the problem having final position and velocity constraints.

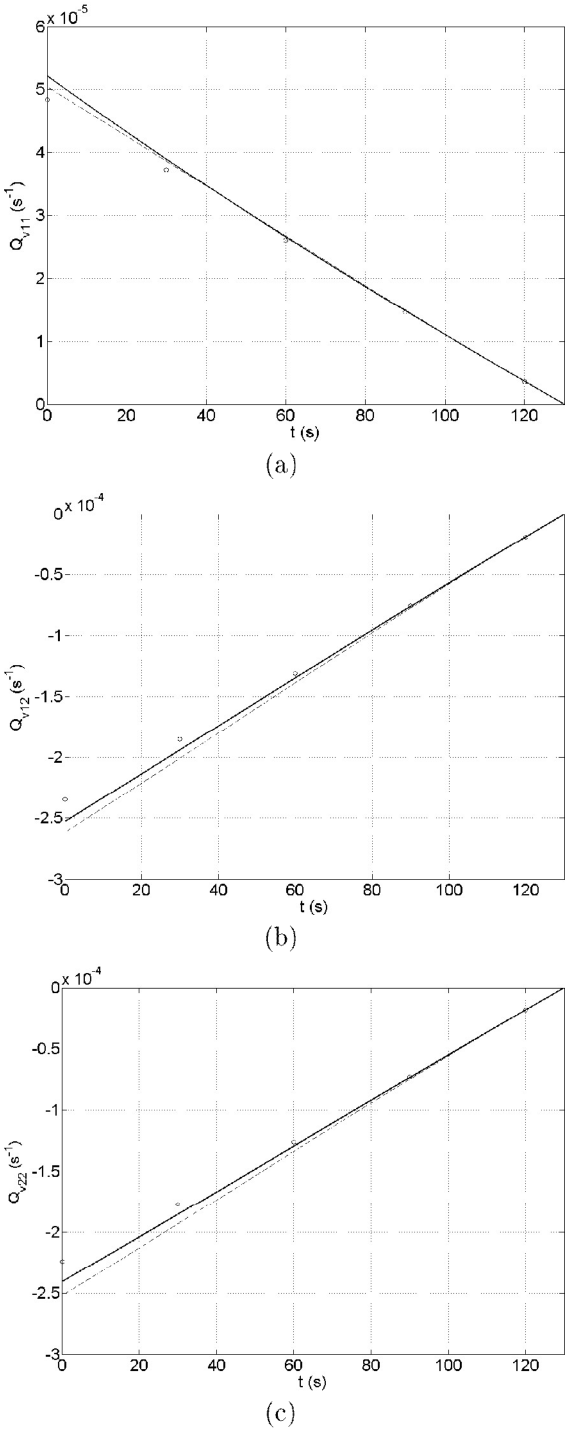

Figure 2. Three different solutions of Q_v elements. Solid line is the solution of differential Eq. (34), dash-line is approximate solution (36), and circles are numerical differentiation using flight simulation.

Substitution of Eq. (40) into the preceding relation yields:

$$\dot{\mathbf{z}}_i = (\dot{F}_i - F_iQ_i)\mathbf{v}_g^i - F_i\,\mathbf{a}_T \qquad (42)$$

If F_i is obtained in such a way that $\dot{F}_i - F_i Q_i = 0_{3\times 3}$, we will have:

$$\dot{\mathbf{z}}_p = -F_p \mathbf{a}_T \tag{43a}$$

$$\dot{\mathbf{z}}_v = -F_v \mathbf{a}_T \tag{43b}$$

The preceding formulation reduces onboard computational burden. In addition, it is more appropriate for analytical approaches than Eq. (40).

For the linearized problem, one can obtain:

$$\mathbf{z}_p = \Phi_{12}(t_f, t)\mathbf{v}_g^p \tag{44a}$$

$$\mathbf{z}_v = \Phi_{22}(t_f, t)\mathbf{v}_g^v \tag{44b}$$

Therefore,

$$\dot{\mathbf{z}}_p = -\Phi_{12}(t_f, t)\mathbf{a}_T \tag{45a}$$

$$\dot{\mathbf{z}}_v = -\Phi_{22}(t_f, t)\mathbf{a}_T \tag{45b}$$

Depending on applications, other changes of variables may be useful. For instance, using $F_p = t_{go} I_{3\times 3}$, one can obtain:

$$\dot{\mathbf{z}}_p = \left(Q_p + \frac{1}{t_{go}} I_{3\times 3}\right)\mathbf{z}_p - t_{go}\mathbf{a}_T \tag{46}$$

The parenthetic expression in the preceding relation will be zero for constant gravity model in the vacuum. Therefore, this expression represents any deviation from the constant gravity model.

CONCLUSIONS

The desired or required velocity of a spacecraft is derived for final velocity constraint with a specified final time. The control system dynamics is taken arbitrary order with nonlinear input matrix. The sensitivity matrix of the required velocity with respect to position vector is then obtained. In addition, the differential equation of the sensitivity matrix is derived assuming that the state variables of the control system are not dependent on the vehicle position and velocity. This sensitivity matrix, besides the conventional Q guidance relation, can be utilized for an implicit guidance of a spacecraft with final position and velocity constraint. For this purpose, the required velocity is defined for the vacuum, but it works in the atmosphere for the ascending phase provided that the cut off (or burnout) occurs in the vacuum. The solution is also obtained in the presence of aerodynamical force. This formulation may be utilized for a trajectory optimization of a spacecraft. Finally, a change of variable is utilized whose benefits are two folded. First, it reduces onboard computational burden. Second, it is more appropriate for analytical approaches than conventional implicit equations.

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