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RESEARCH NOTE

The Comparison of Direct and Indirect Optimization Techniques in Equilibrium Analysis of Multibody Dynamic Systems

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The present paper describes a set of procedures to solve nonlinear static-equilibrium problems in the complex multibody mechanical systems. To find the equilibrium position of the system, five optimization techniques are used to minimize the total potential energy of the system. Comparisons are made between these techniques. A computer program is developed to evaluate the equality constraints and objective function of a general multi-body dynamic system in order to find the equilibrium condition. The obtained results from different approaches are compared together and finally some conclusions are made considering the existing results of the artificial damping method. It is seen that the indirect methods may produce more accurate results with faster convergence.

INTRODUCTION

A mechanism is a multibody system whose components have rigid-body displacements [1]. The kinematic and dynamic analyses involve initial position, static equilibrium position, and also direct and inverse problems in the case of open-or mixed-chain mechanisms [2-7].

For a mechanism with a large number of degrees of freedom, there are infinite possible solutions to the inverse position problem, from among which the most suitable one must be sought on the basis of optimization criteria [8]. The methods used to resolve the position problems, mainly the problem of deformed position and that of static equilibrium with large displacements, are also useful in resolving the inverse problem of open loop mechanisms, as it is described in Reference [9].

In this study, the procedure used to model the mechanisms is basically the one described in [2], which is based on the computational multibody dynamic modeling techniques. Hence, the subsequent mathematical treatment is based on the Jacobian of constraint and the gradient of objective function. Also the related constraints are of the types of the kinematics constraints.

In order to find the equilibrium position of a multibody system, one can use several methods. As the first approach, the equilibrium analysis is carried out normally by adding artificial damping (AD) and solving the equations of motion until the system comes to rest. The results of AD method are relatively more precise than the others and can be used as a comparison criterion, but this method is computationally expensive and time consuming. However, this method cannot be used for real-time assembling and initial position problems. The second widely used approach employs optimization techniques for minimization of Total Potential Energy (TPE) of conservative mechanical systems [3]. In this study three direct optimization techniques namely Sequential Linear Programming (SLP), Modified Method of Feasible Directions (MMFD), and Sequential Quadratic Programming (SQP) and two indirect methods namely Fletcher-Reeves (FR) method and Broyden-Fletcher-Goldfarb-Shanno (BFGS) method, are used.

In order to compare these methods together three test cases are selected. These cases involve three general mechanisms, namely dual pendulum, slider-crank and four bar mechanism. The obtained results

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are compared with each other and with the results obtained by artificial damping approach in order to select the best method for equilibrium analysis.

KINEMATIC AND DYNAMIC MODELING OF MULTIBODY SYSTEMS

A rigid body is defined as a system of particles whose relative distances are forced to remain constant during the motion. However like the approximation of a rigid body as a particle, this assumption could not be strictly true at all. In fact, all bodies deform as they move. However, the approximation remains acceptable as long as the deformations are negligible relative to the overall motion of the body. Therefore, the concept of rigid body elements should be applicable to most of the popular practical mechanisms.

Normally each particle in a rigid body is located by a constant position vector in a reference frame which is attached to the body and moves along with it. This point can be located in the global reference frame as follows;

$$\bar{r} = \bar{r}_0 + A\bar{s} \tag{1}$$

where as shown in Figure 1, $\bar{r}$ and $\bar{r}_0$ are the position vectors that locate a particle and the center of mass of the body with reference to the origin of the inertial frame. Also $\bar{s}$, represents the position vector of the point in the local body-fixed frame and A is the rotation transformation matrix, which for a planar system is as fallows;

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{2}$$

Note that in this case $\bar{r}_0$ and ϕ are generalized coordinates that locate the body in the plane.

A mechanical system or a mechanism is normally considered as a set of rigid bodies which are connected by joints, influenced by forces, driven by prescribed motions, and restricted by constraints. To specify the configuration of a mechanism with nb rigid members at any moment, a generalized coordinate system is chosen as follows;

$$\bar{q} = \begin{bmatrix} \bar{q}_1^T & \bar{q}_2^T & \dots & \bar{q}_{nb}^T \end{bmatrix}^T \tag{3}$$

where $\bar{q}_i$ is the vector of the generalized Cartesian coordinates for body i with dimension nd. For a planar system, this vector can be written as;

$$\bar{q}_i^T = \begin{bmatrix} x_i & y_i & \phi_i \end{bmatrix} \tag{4}$$

The physical conditions imposed by the joints can be expressed as algebraic equations in terms of generalized coordinates and called the vector of holonomic kinematic constraint equations and can be written as;

$$\bar{\phi}(\bar{q}) = \begin{bmatrix} \phi_1(\bar{q}) & \phi_2(\bar{q}) & \dots & \phi_{nc}(\bar{q}) \end{bmatrix}^T \tag{5}$$

where nc is the number of constraint equations. The holonomic constraint equations can be constructed by expression of the imposed conditions between any pair of bodies connected by joints; i.e. in algebraic form and in terms of generalized coordinates for any type of joints like revolute, translational, cam-follower, etc. Furthermore, the Jacobin matrix of the constraints can be assembled in a systematic way for each of them. This matrix that is denoted by $\phi_{\bar{q}}(\bar{q})$ plays an important role in the following theory and numerical methods for finding the equilibrium position of each mechanism.

Combining the Newton-Euler form of constrained equations of motion with the kinematic (or constraints) acceleration equations, one can arrive at a set of algebraic and deferential equations that governs motion of the system as follows:

$$\begin{bmatrix} M & \phi_{\bar{q}}^T \\ \phi_{\bar{q}} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\bar{q}} \\ \bar{\lambda} \end{Bmatrix} = \begin{Bmatrix} \bar{g} \\ \bar{\gamma} \end{Bmatrix} \tag{6}$$

where M is the system mass matrix, $\bar{g}$ is the generalized force vector, $\bar{\lambda}$is the Lagrange multipliers vector, and $\bar{\gamma}$ can be obtained by twice differentiation of the constraint equations as follows [2, 3]:

$$\bar{\gamma} = -(\phi_{\bar{q}}\dot{\bar{q}})_{\bar{q}}\,\dot{\bar{q}} - 2\phi_{\bar{q}t} - \phi_{tt} \tag{7}$$

The Lagrange multiplier form of the equations of motion which will be accepted (Eq. 10) is a mixed system of differential-algebraic equations that must be solved to determine the dynamic motion of the system.

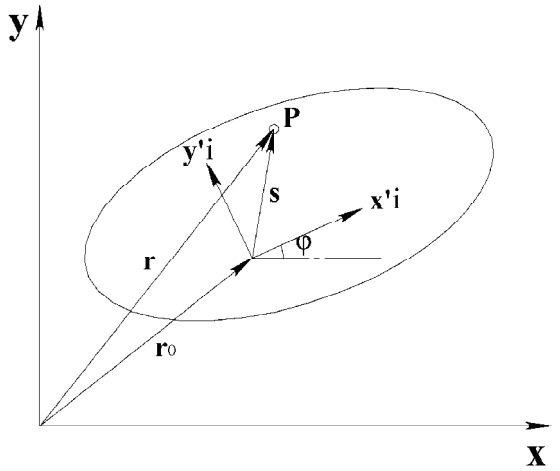

Figure 1. Relation of body and inertial coordinate of a point.

Equilibrium Analysis

A special analysis of a mechanical system that seeks to find a configuration with zero velocity and acceleration is called Equilibrium Analysis (EA). There are three

methods for determining the equilibrium configuration. The first of them which is called dynamic settling involves integration of the equations of motion of the system in time under the action of applied forces, and by addition of artificial damping, until the system comes to rest at an equilibrium configuration. It is worth noticing that for such situations this method is the only valid method for EA, but in general, this method is computationally expensive and time consuming [4]. The second approach, which is called the equations of equilibrium method, is based on the construction of the equations of motion by assuming zero velocity and acceleration and also solving the obtained equations for an equilibrium configuration. This method is not frequently used due to some difficulties in numerical solution arising from ill-conditioning of the problem and existence of unstable configurations. The third method is based on the principle of minimization of Total Potential Energy (TPE) for conservative systems. This principle states that a conservative system is in stable equilibrium condition if and only if the total potential energy is at a strict relative minimum. This method is computationally efficient by using alternate minimization techniques. In this research work the first and third methods are used to find the equilibrium configuration of multibody systems.

The total potential energy of a system involves two terms; the potential of the set of conservative external forces which are assumed to be concentrated at the center of gravity of the members and the elastic potential of deformable elements.

In general, the related optimization problem can be stated as follows:

Find $\bar{q}$ which minimizes $U(\bar{q})$, subject to:

$$\bar{\phi}_k(\bar{q}) = 0, \qquad k = 1, ..., l \tag{8}$$

The most efficient classical optimization methods need to calculate the first and in some cases the second derivatives of the objective functions (TPE) with respect to the generalized coordinate.

In the theory of conservative force systems, the gradient of total potential energy should equalize the negative of the generalized applied forces. i.e.:

$$\bar{U}_{\bar{q}}^T = -\bar{Q}^A \tag{9}$$

Now having the constraints vector, the Jacobian matrix of constraints, and the gradient of objective function, one can use most of the constrained optimization techniques for finding the equilibrium configuration of any mechanical system.

OPTIMIZATION TECHNIQUES

The equilibrium analysis of a multibody mechanism leads to a constrained optimization problem (with

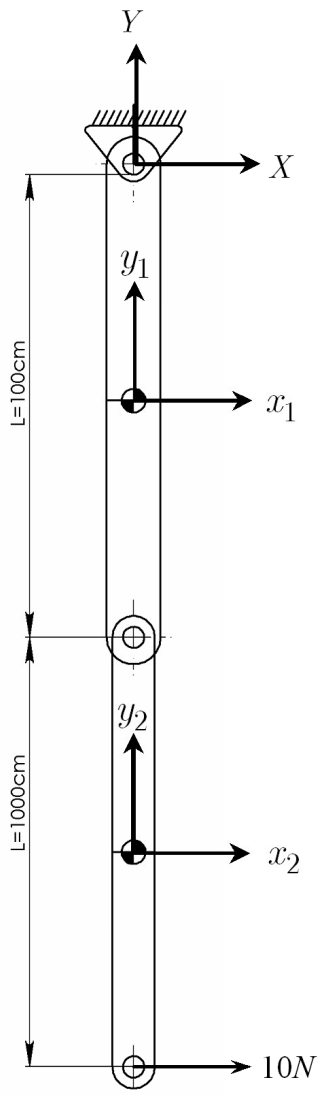

Figure 2. Dual pendulum.

equality constraints) which minimizes the TPE of the system. In general, assuming equality and inequality constraints, the optimization problem can be stated as follows:

Find $\bar{q}$ which minimizes $U(\bar{q})$, subject to:

$$\begin{cases} G_J(\bar{q}) \le 0 & j = 1, ..., m \\ \phi_K(\bar{q}) = 0 & k = 1, ..., l \end{cases} \tag{10}$$

All the methods, which are available for the solution of such problems, can be classified into two broad categories; direct methods and indirect methods. In direct methods constraints are handled in an explicit manner whereas in most of the indirect methods the constrained problem is solved as a sequence of unconstrained optimization problem.

Direct Solutions

In this study, three different methods are used for direct solution, which are Sequential Linear Pro-

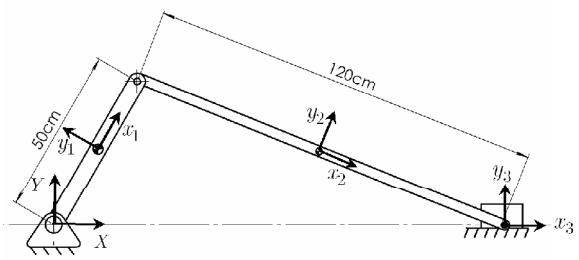

Figure 3. Slider-crank mechanism.

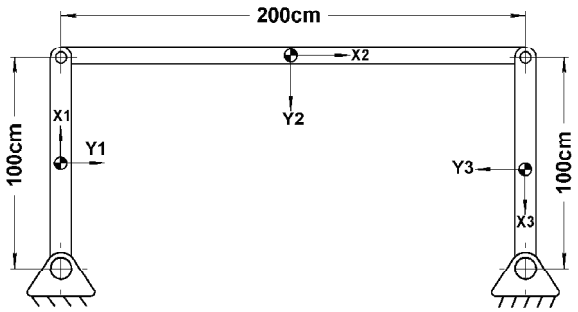

Figure 4. Four-bar mechanism.

gramming (SLP), Modified Method of Feasible Directions (MMFD) and Sequential Quadratic Programming (SQP). In SLP method, one should start with an initial point ($\bar{q}_1$) (which does not need to be feasible) and solve a series of linear programming problems by approximating the nonlinear objective and constraint functions through the first-order Taylor series expanded around the current variable vector ($\bar{q}_i$) [10, 11]. The resulting LP problem is solved using the simplex method to find the new variable vector, ($\bar{q}_{i+1}$). The SLP method is an efficient tool for solving convex programming problems with nearly linear objective and constraint functions. Each of the approximating problems will be a LP problem and hence can be solved quite efficiently

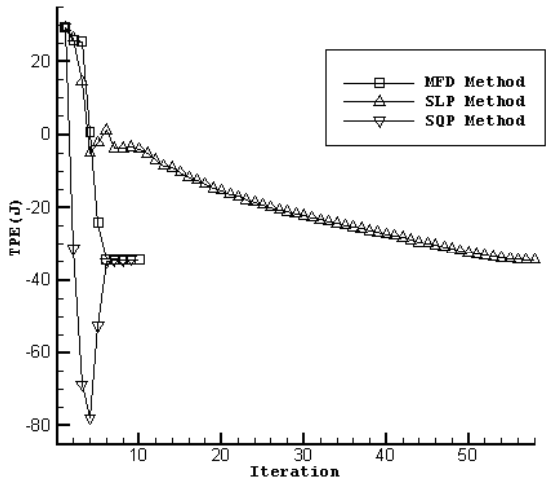

Figure 5. TPE V.S program iteration for direct methods for the Dual pendulum case.

as the dual simplex method can be used. However, this method may suffer the same limitations as the simplex method. Prior information during the search is not used in the subsequent iterations and, therefore, the convergence rate slows down. For the EA of mechanisms due to the use of simplex method, poor search directions may be defined because of the high nonlinear behavior of the objective function. Search success is highly dependent on choice of the pattern of the move limits which is relatively time consuming [10].

In MMFD method we choose a feasible starting point which satisfies all the constraints and moves to the new point according to the iterative scheme of $\bar{q}_{i+1} = \bar{q}_i + \lambda_i \bar{S}_i$, where $\bar{S}_i$ is the direction of movement and λ is the step length. The search direction $\bar{S}_i$ is found such that a small move in that direction violates none of the constraints (feasible) and the value of objective function can be reduced in that direction (usable). This method is not suitable for equality constraints and is computationally expensive due to the addition of the slack variables [10]. One of the best Nonlinear Programming (NLP) methods is the SQP method. The quadratic programming problem involves minimization of a quadratic function subject to linear constraints. The theoretical base of this method is related to the solution of a set of nonlinear equations using the Newton's method and derivation of simultaneous nonlinear equations by applying the Kuhn-Tucker conditions to the Lagrangian form of the constrained optimization problem [10]. The corresponding Lagrange function of the problem has the following form:

$$L(\bar{q}, \gamma) = V(\bar{q}) + \sum_{j=1}^{m} \lambda_j G_j(\bar{q}) + \sum_{k=1}^{l} \lambda_{k+m} \phi_k(\bar{q}) \qquad (11)$$

Therefore, the original problem can be solved iteratively by solving the quadratic programming problem in each step.

Indirect Methods

In these methods unconstrained optimization techniques are used; nevertheless, it is necessary to convert the problem to an unconstrained one. For a mechanical system, we have the vector of nc generalized coordinate system q, and the vector of nh independent constraint equations $\phi(q) = 0$. Reference [3] presents a numerical method for elimination of the dependent variables.

All the unconstrained optimization methods are of iterative type, which start from an initial trial solution and proceed toward the optimum point in a sequential manner. All these methods require an initial point and the difference between them arises from the way that the new point in each computational step is generated. Here, two types of these methods, namely, Fletcher-Reeves (FR) method, and Broyden-Fletcher-Goldfarb-Shanno (BFGS) method are used. In the

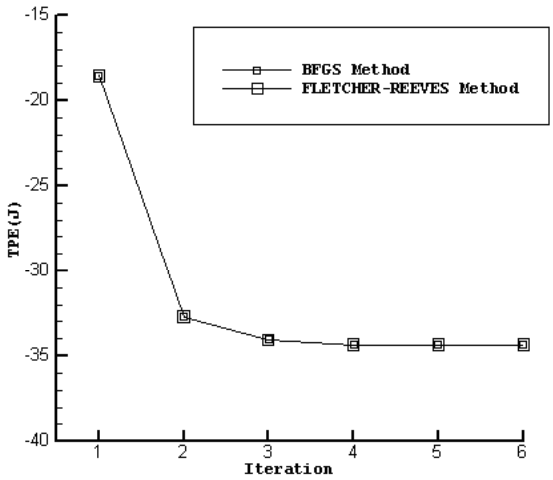

Figure 6. TPE V.S program iteration for indirect methods for the Dual pendulum case.

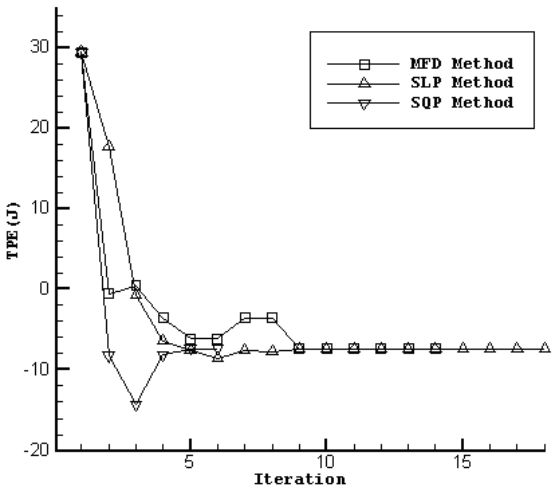

Figure 7. TPE V.S program iteration for direct methods for the slider-crank case.

Fletcher-Reeves method, the first search direction is obtained from $S_1 = -\nabla f(X_1)$ and the new point is found from $X_{i+1} = X_i + \lambda_i^* S_i$, where λ_i^* is the optimal step length in the direction S_i. Also S_i for i>1 is evaluated from $S_i = -\nabla f_i + \frac{|\nabla f_i|^2}{|\nabla f_{i-1}|^2} S_{i-1}$. Note that for a quadratic function, this method converges in n cycles or less, where n is the number of elements of the vector of the generalized coordinate system. But, for ill-conditioned quadratics, the method may require much more than n cycle and to reduce the error it is necessary to restart the method periodically after any m step, where it is an optional value.

In the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method, one starts with an initial point and an $n \times n$ positive definite symmetric matrix [B] as an initial estimate of the Hessian matrix of the objective function, which is usually taken as the identity matrix [I]. The search direction in this method

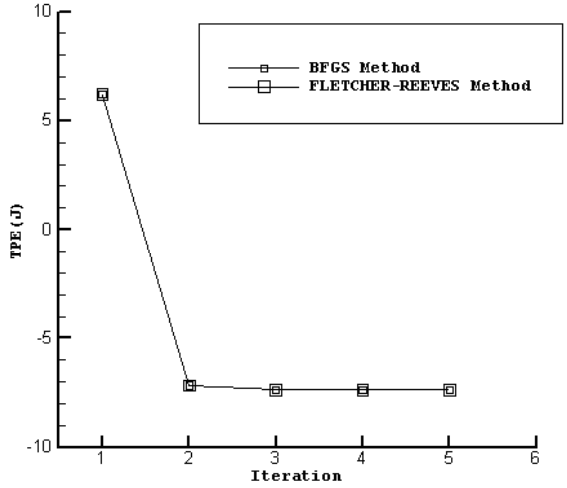

Figure 8. TPE V.S program iteration for indirect methods for the slider-crank case.

is obtained from $S_i = -[B_i]\nabla f(X_i)$ and the new point from $X_{i+1} = X_i + \lambda_i^* S_i$. New search directions are obtained by updating the Hessian matrix as $B_{i+1} = B_i + \left[1 + \frac{g_i^T B_i g_i}{d_i^T g_i}\frac{d_i d_i^T}{d_i^T g_i} - \frac{d_i g_i^T B_i}{d_i^T g_i} - \frac{B_i g_i d_i^T}{d_i^T g_i}\right]$, where $d_i = \lambda_i^* S_i$ and $g_i = \nabla f_{i+1} - \nabla f_i$.

If λ_i^* are found accurately, the matrix $[B_i]$ retains its positive definiteness. Otherwise, it might become indefinite or singular. Therefore, the matrix $[B_i]$ may need to be reset to the identity matrix periodically.

TEST CASES AND DISCUSSION
TEST CASES AND DISCUSSIONS

A set of examples are used to evaluate the performance of the mentioned minimization methods for equilibrium analysis. Three classical test cases are presented. A combination of a general multibody dynamic solver and an optimization code is developed for this purpose. Also, the results of the artificial damping (AD) method are found using this dynamic solver.

The entries of the dynamic code are mass properties of the members, geometric parameters of the mechanism, constraints information, specifications of the spring-damper-actuator sets, and external applied loads. The mass properties of the mechanisms are given in Table 1. Since the starting point in MMFD method must be feasible, a procedure is developed to find a feasible starting point. The convergence criteria

Table 1. Mass properties of test cases.

Case	Member	Mass (Kg)
1	1	1
	2	2
2	1	2
	2	1
	3	1
3	1	1
	2	2
	3	3

Table 2. Equilibrium analysis results. Case 1.

Member	Coord	Direct Optimization techniques			Indirect Optimization techniques		AD
		MFD	SLP	SQP	FR	BFGS	
1	X	0.005	0.012	0.00	0.00	0.00	0.00
	Y	-0.502	-0.500	-0.500	-0.50	-0.50	-0.50
	ϕ	-89.426	-88.610	-89.902	-89.925	-90.024	-90.00
2	X	0.010	0.034	0.003	0.001	0.001	0.00
	Y	-1.505	-1.500	-1.501	-1.50	-1.50	-1.50
	ϕ	-88.905	-88.855	-89.815	-89.821	-89.845	-90.00
Function calls:		220	174	101	33	37	

Table 3. Equilibrium analysis results. Case 2.

Member	Coord	Direct Optimization techniques			Indirect Optimization techniques		AD
		MFD	SLP	SQP	FR	BFGS	
1	X	0.016	-0.001	0.000	0.00	0.00	0.00
	Y	-0.252	-0.250	-0.250	-0.25	-0.25	-0.25
	ϕ	-86.440	-90.276	-89.969	-89.887	-89.887	-90.00
2	X	0.575	0.544	0.546	0.545	0.545	0.545
	Y	-0.252	-0.250	-0.250	-0.25	-0.25	-0.25
	ϕ	24.887	24.674	24.624	24.624	24.624	24.624
Function calls:		321	370	125	20	20	

parameters and the initial values are the same for all direct and indirect methods. The differences between the results of equilibrium points and the number of function calls are used for the sake of comparison between the applied methods. Also, the plots of TPE versus program iteration for each one of the test cases are presented in Figures. 5-10. The first test case is a dual pendulum as it is presented in Figure 2. The results of the equilibrium analysis of this test case and their comparison with the analytical solution are given in Table 2. The second test case is a slider-crank mechanism shown in Figure 3. The results of the equilibrium analysis for this test case are also shown in Table 3. The third case shown in Figure 4 is a four-bar mechanism. Table 4 gives the results of the equilibrium analysis for this test case. It can obviously be seen that by using the direct methods the best results were obtained with the SQP technique. Also, the results show that this method is considerably faster and more precise than the two other methods. Also it can be concluded that the indirect methods are superior to the

Table 4. Equilibrium analysis results. Case 3.

Member	Coord	Optimization techniques			Indirect Optimization techniques		AD
		MFD	SLP	SQP	FR	BFGS	
1	X	-0.310	0.437	-0.042	0.00	0.00	0.00
	Y	-0.389	0.244	-0.499	-0.50	-0.50	-0.50
	ϕ	231.375	29.141	-15.084	-89.977	-89.977	-90.00
2	X	0.216	1.559	0.917	0.10	0.10	0.10
	Y	-0.226	-0.242	-0.997	-0.10	-0.10	-0.10
	ϕ	33.439	-46.793	-18.849	0.00	0.00	0.00
3	X	1.525	2.122	1.958	0.20	0.20	0.20
	Y	0.163	-0.486	-0.498	-0.10	-0.10	-0.10
	ϕ	-18.923	104.046	13.565	-90.00	-90.00	-90.00
Function calls:		271	186	163	34	34	

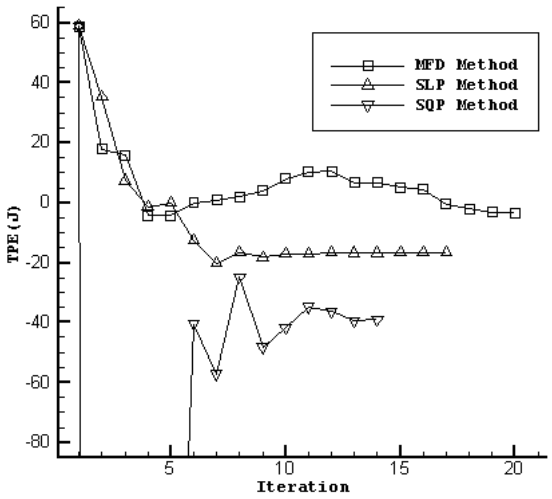

Figure 9. TPE V.S program iteration for indirect methods for the four-bar case.

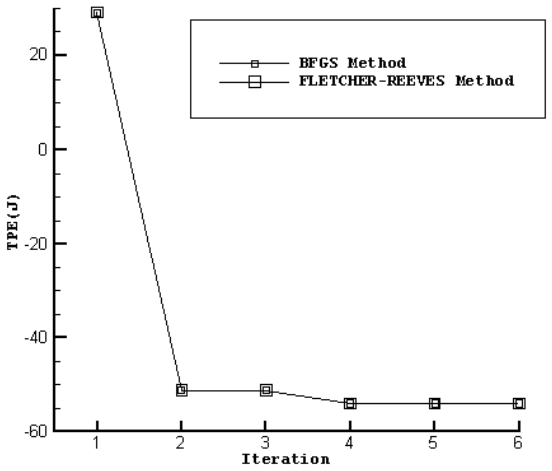

Figure 10. TPE V.S program iteration for indirect methods for the four-bar case.

direct methods in terms of accuracy and the number of function evaluation. Note that the results of the third test case show that the direct methods do not converge while; the indirect methods converge with slower convergence rate in comparison with previous test cases.

CONCLUSIONS

This paper presents a comparison of different approaches to the nonlinear problem of equilibrium analysis in multibody dynamic systems. For this purpose numerous nonlinear direct and indirect optimization techniques have been applied to the solution of these types of problems. The total potential energy of the multibody system is defined as the objective function for minimization. Three direct methods namely, SLP, SQP, and MMFD and two indirect methods namely, FR and BFGS were used. Also using the dynamic solver without optimization techniques, the artificial damping method was used in order to find the equilibrium point and serve as a base for comparison. Several examples have been tested in order to compare the different optimization methods with each other and with the artificial damping method. They involve various simple and complex planar mechanical systems. Because of high nonlinearity of the objective function and constraint equations, the best results, were obtained from the indirect methods with faster convergence rate. Among direct methods the best results were obtained from the SQP method although the SLP method produces acceptable results, especially for simple mechanisms.

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