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Nonlinear Optimal Control Techniques Applied to a Launch Vehicle Autopilot

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This paper presents an application of the nonlinear optimal control techniques to the design of launch vehicle autopilots. The optimal control is given by the solution to the Hamilton-Jacobi-Bellman (HJB) equation, which in this case cannot be solved explicity. A method based upon Successive Galerkin Approximation (SGA) is used to obtain an approximate optimal solution. Simulation results involving the three degrees-of-freedom (3DOF) model of a launch vehicle during atmospheric flight are reported to demonstrate the performance of the considered autopilot. Alternative results are also presented for classical and linear optimal approaches.

NOMENCLATURE

ω_Z	pitch rate
ϑ	pitch angle
$\theta = \vartheta - \alpha$	path angle
δ_ϑ	deflection of control motor
I_Z	pitch moment of inertia
m	mass of launch vehicle
P	thrust force
S	reference area
ρ	density of atmosphere
C_{xa}, C_{ya}^{α}	aerodynamic coefficients
$C_{\vartheta\delta} = -L_P P_{con}$	dynamic coefficient
P_{con}	sum of two control motors thrust
L_P	distance between center of mass and point that control force, (P_{con}) effects
x_P	distance between center of mass and center of pressure
V_{cx}, V_{cy}	Components of launch vehicle velocity

INTRODUCTION

The flight control systems are generally designed using linearized models of nonlinear equations of motion and aerodynamic forces and moments. This requires the flight control system to guarantee stability and performances in the presence of large aerodynamic variations, neglected dynamics and non-linearities [1]. The dynamics of launch vehicles are inherently nonlinear due to inertial coupling, gravitational forces, aerodynamic effects and actuator limits. Much of the recently published launch vehicle control literature focuses on the application of linear optimal control methods to linearized launch vehicle models [2]. Launch vehicle systems often operate in flight regimes where nonlinearities significantly affect dynamic response. Such anticipated performance improvements have motivated research into nonlinear control design techniques.

There are a number of design methods developed for the control of nonlinear systems. A nonlinear Dynamic Inversion controller has been used for an air-to-air missile in Reference [3]. In Reference [4] the State Dependent Riccati Equation (SDRE) has been used for longitudinal missile autopilot design. Reference [5] has proposed a nonlinear design procedure that combines the SDRE and Dynamic Inversion. Five different modern nonlinear control methods have been discussed in [6]. In this paper a nonlinear optimal control is designed by solution to the HJB equation. The Successive Galerkin Approximation has been developed to approximate the solution to the HJB equation in a form that is amenable to practical feedback control [7]. The idea of this method is to reduce the nonlinear HJB partial differential equation (PDE) to a convergent sequence of linear PDEs and then to approximate the solution to each PDE via a Galerkin approximation technique. The result is a numerical algorithm that

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computes the coefficients needed for a nonlinear feedback control law. This technique has been applied to a number of practical design problems [2, 7, 8, 9, and 10]. A comprehensive comparison has been carried out in [11] between the SGA method and five other methods for synthesis of nonlinear control systems. In Reference [12] the SGA method has been used for developing tracking controllers for nonlinear systems.

Nonlinear optimal control approach is described in section 2. Launch vehicle equations of motion are detailed in section 3. Control synthesis and implementation are presented in section 4. The simulation results are presented in section 5. A comparison between SGA, classical and linear optimal controllers is also presented in the same section.

NONLINEAR OPTIMAL CONTROL APPROACH

For a system modeled by the nonlinear state equations:

$$x^{\bullet} = f(x) + g(x)u(x), \tag{1}$$

and the performance index:

$$J(x) = \int_0^{\infty} (l(x) + \|u(x)\|^2 R)dt, \tag{2}$$

where $l(x)$ can be any positive definite function of x, commonly in the form $x^T Qx$, which Q is a semi-positive definite matrix and R is a positive definite matrix [1, 10].

The optimal control which minimizes the performance index is given by:

$$u^*(x) = -\frac{1}{2}R^{-1}g^T(x)\frac{\partial J^*}{\partial x}, \tag{3}$$

where $J^*(x)$ is a positive definite function that satisfies the Hamilton–Jacobi– Bellman (HJB) equation [13]:

$$\frac{\partial J^{*T}}{\partial x}f(x) + l(x) - \frac{1}{4}\frac{\partial J^{*T}}{\partial x}g(x)R^{-1}g^T(x)\frac{\partial J^*}{\partial x} = 0, \tag{4}$$

If system (1) is linear and the state penalty function $l(x)$ is quadratic the HJB equation reduces to the standard Riccati equation of optimal control [7, 13].

Successive Approximation

This method that is sometimes called “iteration in policy space” was first used in the context of the HJB equation by Bellman [7, 14].

The HJB equation (3, 4) can be generalized to form an iterative solution process where:

$$\frac{\partial J^{(i)T}}{\partial x}\left[f + gu^{(i)}\right] + l + \left\|u^{(i)}\right\|^2 R = 0, \tag{5}$$

$$u^{(i+1)} = -\frac{1}{2}R^{-1}g^T\frac{\partial J^{(i)}}{\partial x}. \tag{6}$$

Equation (5) is known as the Generalized Hamilton–Jacobi–Bellman (GHJB) equation. Starting with a known control, $u^{(0)}$ (e. g. PD control) that stabilizes over a bounded domain of the state space Ω, and iteratively solves Equations (5) and (6), successive value for $J^{(i)}$ and $u^{(i)}$ are found which have been shown to converge to the optimal values u^* and J^* given by Equations (3) and (4) respectively. This method reduces the HJB equation to an infinite sequence of linear partial differential equations of the form (5) where u is a known function of x. This equation is a linear first–order partial differential equation [7]. One can make the following statements about the sequence $\{J^{(i)}\}_{i=0}^{\infty}$ and $\{u^{(i)}\}_{i=0}^{\infty}$ [7]:

- For each $x \in \Omega$:
  $J^*(x) \leq J^{(i+1)}(x) \leq J^{(i)}(x)$,
  if $J^{(i+1)}(x) = J^{(i)}(x)$ then:
  $J^{(i+1)}(x) = J^*(x)$.
- If $u^{(0)}$ stabilizes system (1) on Ω, then so does $u^{(i)}$ for all $i \geq 0$.
- The control $u^{(i+1)}$ has better performance in the same sense as the optimal control, u^*, for example 50% gain reduction margin and infinite gain margin [15,16]. The difficulty is that the GHJB equation is difficult to solve analytically.

Galerkin Approximation

To solve a partial differential equation A(J)=0 with boundary conditions J (0)=0, Galerkin's method assumes that we can find a complete set of basis functions $\{\phi_j\}_{j=1}^{\infty}$ such that $\phi_j(0) = 0$ ($\forall_j$), and $J(x) = \sum_{j=1}^{\infty} C_j\phi_j(x)$, where the sum is assumed to converge point wise in some set Ω. An approximation to J is $J_N(x) = \sum_{j=1}^{N} C_j\phi_j(x)$ and the coefficients C_j are obtained by solving the algebraic equation:

$$\int_{\Omega} A(J(x))\phi_s(x)ds = 0, \qquad S = 1 \text{ to } N \tag{7}$$

The set $\{\phi_j\}_{j=1}^{\infty}$ is a complete basis set for the domain of partial differential equation [7, 17].

Numerical Solution of The Ghjb Equation

In Reference [7] a computational Galerkin method has been proposed to solve GHJB equation (5). It is assumed that:

$$J^{(i)} = \sum_{j=1}^{\infty} C_j^{(i)}\phi_j(x), \tag{8}$$

and that the basis functions, $\{\phi_j(x)\}_{j=1}^{\infty}$ are continuous and defined every where on Ω. An approximation to $J^{(i)}$ can be formed by the first N terms of the infinite series:

$$J_N^{(i)} = \sum_{j=1}^{N} C_j^{(i)} \phi_j(x), \tag{9}$$

where the coefficients $(C_1^{(i)}, C_2^{(i)}, ..., C_N^{(i)})$ are obtained by solving:

$$\int_\Omega \left[\frac{\partial \left(\sum\limits_{j=1}^{N} C_j^{(i)} \phi_j \right)}{\partial x} \left(f + g u^{(i)} \right) + l + \left\| u^{(i)} \right\|^2 R \right] \phi_S dx = 0, \quad S = 1 \text{ to } N \tag{10}$$

These equations can be solved to find $C^{(i)} = \left[C_1^{(i)} C_2^{(i)} ... C_N^{(i)} \right]^T$ and $u^{(i+1)}$ can be calculated from equation (6).

SGA Feedback Synthesis Algorithm
By definition, the inner product of two functions f(x) and g(x) are:

$$< f(x), g(x) >_\Omega = \int_\Omega f(x) g(x) dx. \tag{11}$$

The computation of the coefficients is performed according to the SGA feedback algorithm below [10]:

- Input:

$$f, g, l, R, \Omega, \{\phi_i(x)\}_{i=1}^N, u^{(0)}.$$

- Compute the matrices:

$$A_1 = \begin{bmatrix} < \frac{\partial \phi_1^T}{\partial x} f, \phi_1 >_\Omega & \cdots & < \frac{\partial \phi_N^T}{\partial x} f, \phi_1 >_\Omega \\ \vdots & \ddots & \vdots \\ < \frac{\partial \phi_1^T}{\partial x} f, \phi_N >_\Omega & \cdots & < \frac{\partial \phi_N^T}{\partial x} f, \phi_N >_\Omega \end{bmatrix},$$

$$A_2 = \begin{bmatrix} < \frac{\partial \phi_1^T}{\partial x} g u^{(0)}, \phi_1 >_\Omega & \cdots & < \frac{\partial \phi_N^T}{\partial x} g u^{(0)}, \phi_1 >_\Omega \\ \vdots & \ddots & \vdots \\ < \frac{\partial \phi_1^T}{\partial x} g u^{(0)}, \phi_N >_\Omega & \cdots & < \frac{\partial \phi_N^T}{\partial x} g u^{(0)}, \phi_N >_\Omega \end{bmatrix},$$

$$b_1 = - \begin{bmatrix} < l, \phi_1 >_\Omega \\ \vdots \\ < l, \phi_N >_\Omega \end{bmatrix}, \quad b_2 = - \begin{bmatrix} < \left\| u^{(0)} \right\|^2 R, \phi_1 >_\Omega \\ \vdots \\ < \left\| u^{(0)} \right\|^2 R, \phi_N >_\Omega \end{bmatrix},$$

$$\{M_i\}_1^N = \begin{bmatrix} m_i(1,1) & \cdots & m_i(1,N) \\ \vdots & \ddots & \vdots \\ m_i(N,1) & \cdots & m_i(N,N) \end{bmatrix},$$

$$m_i(j,k) = < \frac{\partial \phi_k^T}{\partial x} g R^{-1} g^T \frac{\partial \phi_i}{\partial x}, \phi_j >_\Omega .$$

- Initial step:

$$A = A_1 + A_2, \quad b = b_1 + b_2,$$

$$C^{(0)} = A^{-1} b, \quad i = 1.$$

- Iteration step:

$$A_2 = -\frac{1}{2} \sum_{j=1}^{N} C_j^{(i-1)} M_j,$$

$$b_2 = -\frac{1}{4} \begin{bmatrix} C^{(i-1)T} M_1 C^{(i-1)} \\ C^{(i-1)T} M_2 C^{(i-1)} \\ \vdots \\ C^{(i-1)T} M_N C^{(i-1)} \end{bmatrix},$$

$$A = A_1 + A_2, \quad b = b_1 + b_2,$$

$$C^{(i)} = A^{-1} b, \quad i = i + 1.$$

- Output:

$$u_N^{(i)} = -\frac{1}{2} R^{-1} g^T \left(\sum_{j=1}^{N} C_j^{(i)} \frac{\partial \phi_j}{\partial x} \right).$$

In [18] conditions under which this algorithm converges to the solution of the Hamilton–Jacobi–Bellman equation were developed, also it was shown that for N sufficiently large, the approximate controls stabilize the system and are robust in the same sense as the optimal control.

LAUNCH VEHICLE NONLINEAR MODEL

The dynamic model of the launch vehicle employed in the present research is from [19]. This model consists of a set of nonlinear differential equations describing the pitch plane rigid–body dynamics of a launch vehicle. The model assumes all the lateral state variables are zero and equations are written in the body frame. The equations of motion are given by:

$$m(\frac{dV_{cx}}{dt} - \omega_z V_{cy}) = m g_x + P + \frac{1}{2} \rho S C_{xa} V_{cx}, \tag{12}$$

$$m(\frac{dV_{cy}}{dt} + \omega_z V cx) = m g_y - \frac{1}{2} (C_{xa} + C_{ya}^\alpha) \rho S V_{cx} V_{cy}, \tag{13}$$

$$I_Z \frac{dw_z}{dt} = \frac{1}{2} x_p (C_{xa} + C_{ya}^\alpha) \rho S V_{cx} V_{cy} + C_{\vartheta\delta} \delta_\vartheta, \tag{14}$$

$$\frac{d\vartheta}{dt} = \omega_Z. \tag{15}$$

All coefficients in these equations are time-variant. For example, the mass decreases during the powered flight at a constant rate, C_{xa} is a function of Mach number and angle of attack.

The control objective is to regulate ϑ to a pitch program or command value ϑ_P. The states are V_{cx}, V_{cy}, ω_z, ϑ and the control is δ_ϑ. The coefficients in equations (12) through (15) are functions of state variables and have time–variant behavior during the powered flight. But in the underlying theory of the SGA, system equations must be time–invariant. To do so, we consider equations (12) through (15) in some sample times and assume that the coefficients are constant in any sample time, and then the controller is designed for these sample times.

In any sample time, the objective is to regulate ϑ to a non-zero value (value of ϑ_P in the corresponding time), to do so, the other states and the control must go to non-zero values, but the underlying theory of the SGA algorithm requires that $J(x)$ be finite, and therefore, the states and controls must go toward zero as time moves toward infinity, and a deviation of variables must be performed to satisfy this requirement [10]. For a given desired nominal pith-angle ϑ_P, the corresponding nominal values for the states can be calculated from the information of nominal trajectory. By performing the deviation of variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V_{cx} - V_{cxss} \\ V_{cy} - V_{cyss} \\ \omega_z - \omega_{zss} \\ \vartheta - \vartheta_p \end{bmatrix},$$

a new set of state equations will be defined where a stabilizing control will cause $\vartheta \to \vartheta_P$, $x \to 0$ and $u \to 0$ as $t \to \infty$. The output of interest is $y = x_4$. Equations (12) through (15) represent the nonlinear dynamic of the system. These equations are of the general form of equation (1), and from them the functions $f(x)$ and $g(x)$ which are required for the synthesis of the control have been determined.

CONTROL SYNTHESIS AND IMPLEMENTATION

In any sample time, with the model described above, the optimal feedback control can be synthesized based on the SGA algorithm. With $f(x)$ and $g(x)$ coming directly from the equations of motion, the cost function on the states $l(x)$, the weighting matrix on the control cost R , the domain of the states Ω , the basis functions $\{\phi_j(x)\}_{j=1}^N$, and the initial control low $u^{(0)}$ remain to be determined. For the results presented here, the cost function was chosen to be:

$$l(x) = x_4^2.$$

Since the system has only a single input (δ_ϑ), the weighting matrix on the control, R is a scalar variable and is set to one. The domain of possible values for the states is determined both by the physical capabilities of the system and the likely deviation of the states from their nominal value of zero. For the results presented here, Ω was defined to be:

$$-5m/s \le x_1 \le +5m/s,$$

$$-5m/s \le x_2 \le +5m/s,$$

$$-60\deg/s \le x_3 \le +60deg/s,$$

$$-10\deg \le x_4 \le +10\deg.$$

The proper selection of basis functions is a critical part in the design of controllers using SGA. The basis functions used determine not only the accuracy of the Galerkin approximation, but also the function of the states from which the control law is calculated. The selection of the basis functions also has implications on the operation of the SGA algorithm. If the basis functions do not approximate the value of the function $J^{(i)}(x)$ with sufficient accuracy, the algorithm will fail to converge. For this problem, a second-order set of polynomial basis functions was used as

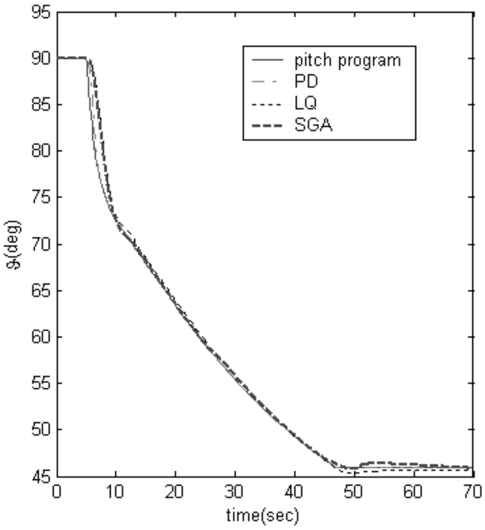

Figure 1. Pitch angle (ϑ).

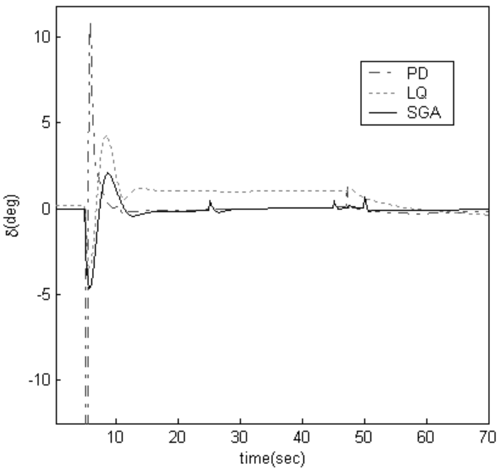

Figure 2. Control motor deflection (δ_ϑ).

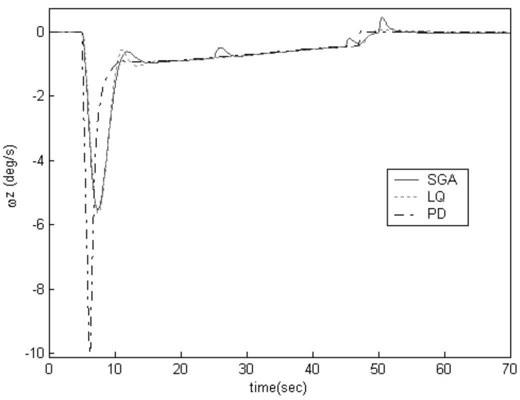

Figure 3. Angular velocity about z-axis

set 1 (default set) [8]:

$$\{\phi_j\}_{j=1}^{10} = \left\{ \begin{array}{c} x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, \\ x_3^2, x_1x_4, x_2x_4, x_3x_4, x_4^3 \end{array} \right\},$$

Second set of basis functions was selected for comparison of simulation results:

$$\{\phi_j\}_{j=1}^{6} = \left\{x_3^2, x_2x_4, x_3x_4, x_4^2, \sin^2 x_4, \tan^2 x_4\right\},$$

The initial stabilizing control has been designed by linearizing the equations of motion. This controller is obtained from solving the Riccati equation in any time. For this problem, the following initial control has been designed in any sample time:

$$u^{(0)} = 5.6x_3 + 8x_4.$$

SIMULATION RESULTS

Using the SGA synthesis algorithm, a nonlinear optimal control law was computed for the pitch channel of the launch vehicle control system. Pitch angle with PD, LQ and SGA controllers is shown in Figure (1). This figure shows that SGA controller has a good performance as linear quadratic and PD controllers and tracks the pitch program exactly. In Figure (2), control motor deflection, δ_ϑ is shown. The deflection of control motor in the classical method is very large. SGA controller is better than PD and linear optimal controllers and has minimum control effort. Pitch rate, ω_Z is presented in Figure (3) and shows that the pitch rate of the vehicle with the SGA controller is less than with other controllers. Angle of attack is presented in Figure (4) for three different controllers during the powered flight. Path angle θ is shown in Figure (5). The effects of choosing different sets of basis functions are presented in Figures (6) and (7). Noisy response of SGA design considering wind perturbation is compared with the LQ controller in Figure (8).

As with the design of a linear quadratic regulator, changes in the behavior of this nonlinear system are brought about by changing the state weighting function $l(x)$ and the weight on the control R.

It should be noted that the SGA design is optimal with respect to the basis functions used in the approximation. It is likely that a different set of basis functions could be chosen that would result in a better approximation. Furthermore, by adding more complex basis functions, higher order nonlinear effects can be compensated [2]. If a standard complete basis (like polynomial or sinusoid) is blindly used in the algorithm, i.e., there is no basis selection, then the amount of memory required to compute and store the coefficients grows exponentially. To overcome this problem, particular basis functions must be chosen that capture the essential dynamics of the problem. However, this creates problems of its own. The selection of the basis

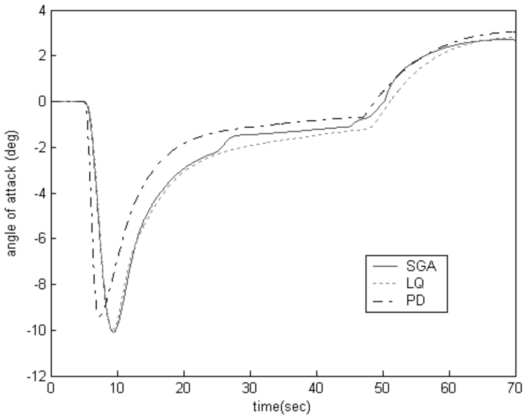

Figure 4. angle of attack (α).

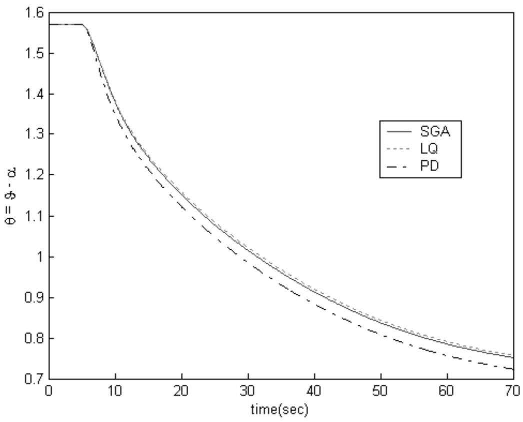

Figure 5. path angle (θ).

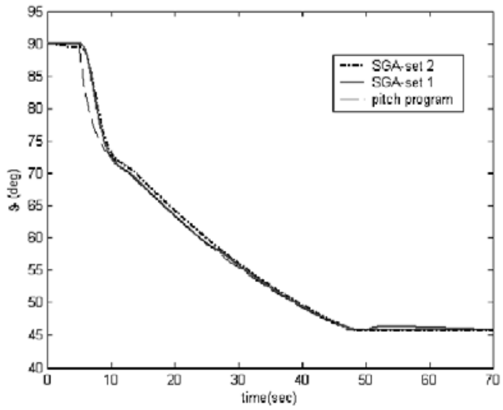

Figure 6. Pitch angle with two different sets of basis functions.

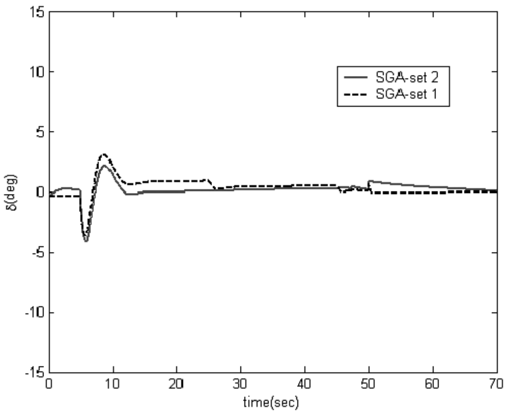

Figure 7. Control motor deflection (δ_ϑ) with two different sets of basis functions.

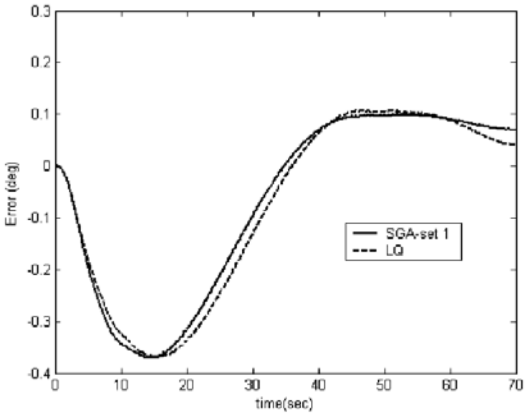

Figure 8. Perturbed error of pitch angle for the SGA and LQ controllers.

functions requires understanding of particular problem being solved. Further research needs to be directed at automating the selection of appropriate basis functions[7]. The resulting approximation scheme has several interesting properties:

- The algorithm must be started with an initial stabilizing control. If the order of approximation is large enough, then one interpretation of the algorithm is that it improves the performance of the control at each iteration.
- Result is closed-loop control that is easy to implement.
- The region (in state-space) of convergence for the approximate control is dictated by the stability region for the initial stabilizing control, and is therefore known a priori and is usually defined explicitly by the designer.
- The stability region of the approximate control is equal to the region of convergence.
- The computations are performed off-line. The on-line burden consists of computing linear combinations of state-dependent basis functions.
- Through judicious selection of the basis functions, the curse of dimensionality can be mitigated [20].
- Once the basis functions are selected, the approximation process can be completely automated.
- The resulting control is given by a linear combination of terms involving a set of basis functions.
- This method is a series-based approximation.
- If a quadratic penalty function is used, then tuning the control becomes computationally simple.
- The robustness properties for nonlinear optimal control are similar as LQR: 50% gain reduction margin and infinite gain margin. Since SGA approximates the nonlinear optimal controller, it will recover the optimal robustness margins as the number of terms in the approximation increases. It should be better than linearizing and applying an LQR controller.

CONCLUSIONS

In this paper, an optimal control method upon successive Galerkin approximation has been applied to the pitch-axis autopilot of a launch vehicle system. This controller is compared with two other controllers. Simulation results demonstrate improvements in control motor deflection of SGA controller over linear optimal control and classical control. Based upon these results, SGA appears to be a promising approach to control synthesis of systems having nonlinear dynamics.

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APPENDIX A

Structure and details of a typical classical LV attitude control system are discussed in [19]. In this problem, pitch angle is considered as input of the system. However the system is time varying and transfer function cannot be defined in the usual way. Thus, the frozen pole approximation is used [19].

It considers that the coefficients of vehicle[s model have constant value during certain interval of time. In this way a transfer function can be obtained. Using well-known methods of classical control theory the control law may be obtained as follow [19]:

$$G_C = K_C(T_C s + 1),$$

The coefficients K_C, T_C are selected for PD controller by using classical frequency response method. In this way PD controller is determined.

A single fourth-order vector matrix equation defining the system is obtained from (12)-(15). Linear optimal controller is designed for any sample time using standard optimal control method presented in [13].

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